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XXXIII. *A new Determination of the Viscosity of Air by the Rotating Cylinder Method. By GUNNAR KELLSTRÖM, D.Sc., Physics Laboratory, University of Uppsala **.

[Plate III.]

IN 1916 Millikan †, using the "oil-drop" method, obtained the value $e=4.774 \times 10^{-10}$ e.s.u. for the electronic charge, later corrected to 4.770×10^{-10} e.s.u. ‡. Absolute wave-length measurements of X-rays have led to a 0.7 per cent. higher value, $e=4.805 \times 10^{-10}$ e.s.u. §. Attempts which have been made to explain this discrepancy as due to errors in the X-ray method || ¶ have proved untenable. A most simple and highly probable explanation of the discrepancy has been suggested by Shiba **, *i. e.*, that Millikan has used somewhat too low

* Communicated by Prof. Manne Siegbahn, D.Sc.

† R. A. Millikan, Phil. Mag. xxxiv. p. 1 (1917).

‡ R. A. Millikan, Phys. Rev. xxxv. p. 1231 (1930); see also R. T. Birge, Phys. Rev. Suppl. I, p. 1 (1929); Phys. Rev. xl. p. 228 (1932).

§ E. Bäcklin, Diss. Uppsala (1928); Zeits. f. Phys. xciii. p. 450 (1934); J. A. Bearden, Phys. Rev. xlvi. p. 385 (1935); J. M. Cork, Phys. Rev. xxxv. pp. 128, 1456 (1930); M. Söderman, 'Nature,' cxxxv. p. 67 (1935); Diss. Uppsala (1935).

|| F. Zwicky, Proc. Nat. Acad. Sc. xvi. p. 211 (1930).

¶ A. W. Porter, Phil. Mag. v. p. 1067 (1928); J. A. Prins, 'Nature,' cxxiv. p. 370 (1929).

** K. Shiba, Sci. Pap. Inst. phys. chem. Res. Tokyo, xix. p. 97 (1932); xxi. p. 128 (1933).

a value of the coefficient of viscosity of air, η , which enters as a factor of the power 3/2 in calculating e by the oil-drop method. The accuracy in e , obtained by this method, will in fact largely depend on the accuracy obtainable in η . Millikan, in 1916, used the value $\eta_{23} = 18226 \times 10^{-8}$, determined by Harrington * in Millikan's laboratory, while he, in 1913, used the value $\eta_{23} = 18240 \times 10^{-8}$ obtained as the mean of five different determinations †. Vogel ‡, in 1914, carried out a similar mean value calculation (27 authors), and got $\eta_0 = 1724 \times 10^{-7}$, corresponding to $\eta_{23} = 1833 \times 10^{-7}$, i. e., 0.57 per cent. higher than Harrington's value. Shiba §, using in addition some new determinations, arrives at $\eta_{23} = 18312 \times 10^{-8}$ as the most probable value, which would give the oil-drop value $e = \left(\frac{18312}{18226}\right)^{3/2} \times 4.770 \times 10^{-10} = 4.804 \times 10^{-10}$, in good agreement with the X-ray value.

Mean value calculations like these must, however, be rather arbitrary, as they comprise determinations of very unequal reliability, often difficult to estimate. Many of these determinations have been made for certain special purposes, such as the variation of viscosity with temperature, the influence of adding other gases, and so on. Such determinations generally cannot claim the same accuracy as those which have been carried out for the special purpose of obtaining η with the greatest possible precision at ordinary temperature and pressure. To the latter sort of determinations belong mainly those which at the initiative of Millikan have been carried out by Rapp || and Markwell ¶, using the capillary tube method, and by Gilchrist ** and Harrington ††, using the rotating cylinder method. Their results show a very good mutual agreement, but are somewhat low compared to the majority of other investigations. It is not improbable that they—in spite of all precautions—may contain some systematic errors, and that the accuracy has been over-estimated. Considering the fundamental importance of

* E. L. Harrington, *Phys. Rev.* viii. p. 738 (1916).

† R. A. Millikan, *Ann. der Phys.* xli. p. 759 (1913).

‡ H. Vogel, *Ann. der Phys.* xlvi. p. 1235 (1914).

§ K. Shiba, *loc. cit.*

|| I. M. Rapp, *Phys. Rev.* ii. p. 363 (1913).

¶ E. Markwell, *Phys. Rev.* viii. p. 479 (1916).

** L. Gilchrist, *Phys. Rev.* i. p. 124 (1913).

†† E. L. Harrington, *loc. cit.*

the constant e I have thought it of interest to carry out a new determination of η by the method that, according to Millikan, permits the highest precision, viz., the rotating cylinder method.

Experimental Method.—The principle of this method, first used by Mallock * and Couette † for liquids, and later by Gilchrist, Timiriazeff ‡, and Harrington for air, is as follows. An inner cylinder is suspended with its axis vertical, and is at the upper and lower end surrounded by guard cylinders of the same diameter. An outer, coaxial cylinder is rotating with constant, low velocity, and produces, through the viscosity of the air, a drag upon the inner cylinder, causing it to be deflected to such an angle from its position of rest that the restoring torsional couple of the suspension is in equilibrium with the drag of the air. Presuming that no end effects appear and that the velocity of rotation is so small that the movement of the air is laminar η can be calculated from the equation §

$$\eta = \frac{I(b^2 - a^2)\phi \cdot t}{2a^2b^2l \cdot T^2}, \dots \quad (1)$$

where ϕ is the angle of deflexion of the inner cylinder, a , l , I , and T its radius, height, moment of inertia, and period of oscillation, b the inner radius of the outer cylinder, and t the time of revolution of the outer cylinder. The equation may be written

$$\eta = K \frac{\phi \cdot t}{T^2}, \dots \quad (2)$$

where

$$K = \frac{I(b^2 - a^2)}{2a^2b^2l}. \dots \quad (3)$$

The instrument constant K is determined once for all and every individual determination of η then includes a determination of ϕ , t , and T .

The Apparatus.—The apparatus used in this determination was made at the workshop of this Institute.

* A. Mallock, Proc. Roy. Soc. xlv. p. 126 (1888).

† M. Couette, Ann. chim. phys. xxi. p. 433 (1890).

‡ A. Timiriazeff, Ann. d. Phys. xl. p. 971 (1913).

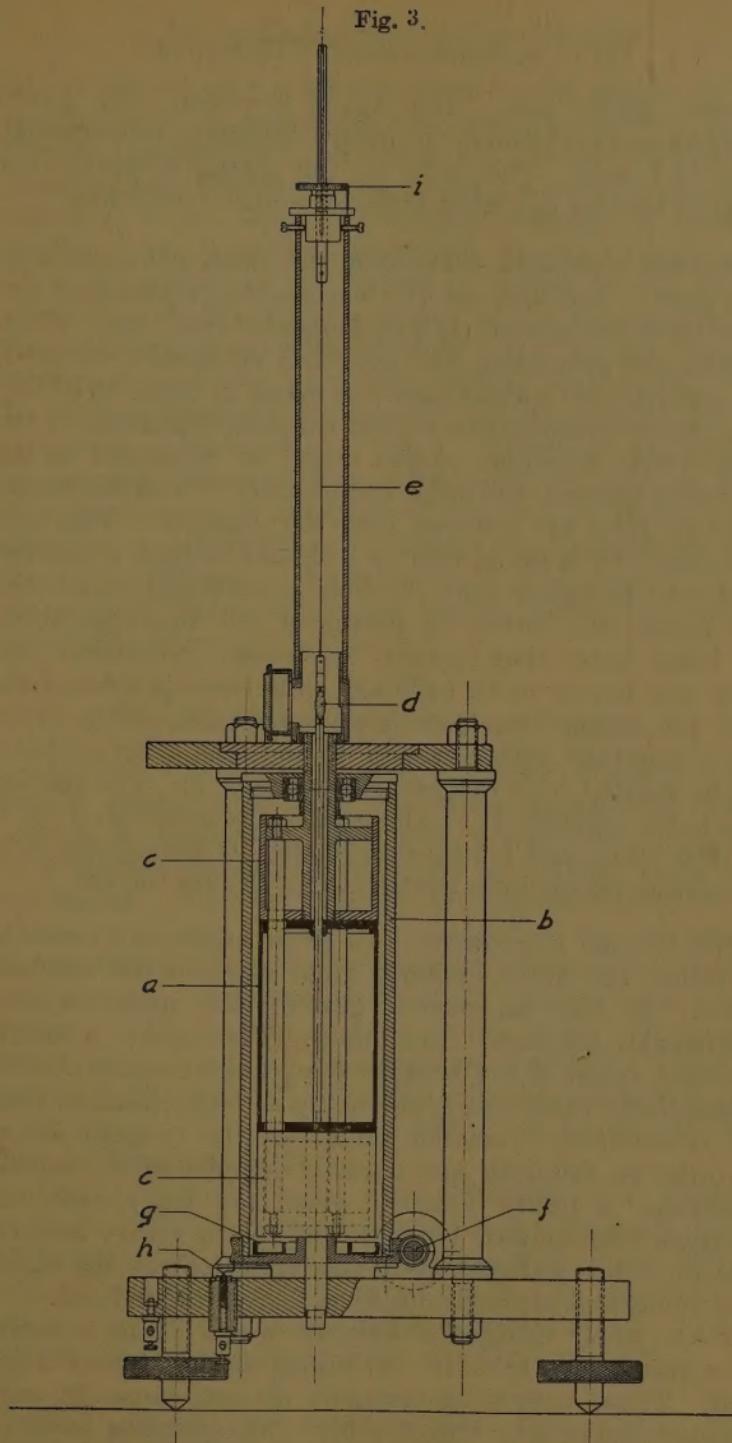
§ See Poynting and Thomson, 'A Textbook of Physics.—I. Properties of Matter,' p. 212, or G. F. C. Scarle, 'Experimental Physics' (Cambridge, 1934), p. 246.

Its appearance and construction are seen from the photographs (Pl. III. figs. 1 & 2) and fig. 3. It was given considerably less dimensions than the Gilchrist-Harrington apparatus; in that way greater precision in manufacture and more convenient adjustment and measuring should be attainable without impairing the relative accuracy.

The Inner Cylinder (fig. 3, *a*), which should have as small a weight and moment of inertia as possible, is cast of electron metal (sp. gr. 1.8). The bottom is cast in one piece with the cylinder, the top screwed into the cylinder, after the inner cylinder surface has been fashioned. The outer cylinder surface is ground to an exact circular cylinder in the grinding machine. The thickness of the walls is about 1.5 mm. and of the ends about 6 mm. The weight is 66.7 gr. In each end there are three small sections cut out for the three steel rods, which hold the guard cylinders together, as well as accurately centred holes for the 24 cm. long central brass rod, whose lower end is screwed into the bottom of the cylinder. Its upper end is conically ground and has an arrangement for fastening the suspension and adjusting it along the axis of the rod. It also carries the mirror.

The Guard Cylinders (fig. 3, *cc*) are held together by three ground steel rods, fitting very accurately into the corresponding holes in the ends of the cylinders. The upper guard cylinder is carried by a brass tube, the upper end of which is screwed into a circular brass disk, accurately fitting in a hole in the upper plate of the tripod stand. Another 25 cm. long brass tube, which carries the suspension for the inner cylinder, is screwed on to the upper end of this brass tube. In the axis of the lower guard cylinder a steel pin with 10 mm. diameter is fixed. This provides the lower bearing for the outer cylinder, and fits into a hole in the base plate of the stand. The whole guard cylinder system is ground in the grinding machine after being put together, so that the axes of the cylinders, the brass tube, and the steel pin really coincide. The steel rods fit so accurately in their holes that the upper guard cylinder can be taken off and put on without disturbing the centering. The distance between the guard cylinders is 100.20 mm. and the height of the inner

Fig. 3.



Drawing of the apparatus. a = inner cylinder, b = outer cylinder, c = guard cylinders, d = plane mirror, e = suspension, f = tangent screw, g = ring-shaped dish for P_2O_5 , h = mercury contact, i = screw for adjusting height of inner cylinder.

cylinder 99.98 mm. The space between the guard cylinders and each end of the inner cylinder is consequently about 0.1 mm. The radii of the guard cylinders are equal to that of the inner cylinder within 0.003 mm.

The Outer Cylinders.—Two different outer cylinders have been used. The first one (I.) was made of a standard size brass tube, the second (II.) of compact brass bolt. When turning and grounding the cylindric surfaces (cylinder I. was not ground) special care was taken in order to obtain an accurate cylinder shape with constant thickness of the wall, so that the inner radius might be measured as the difference between the outer radius and the wall thickness. The two ends are screwed into the cylinder. They are "steered" by a lip as well as a thread in such a manner as always to insure that the hole is accurately centered. The upper end moves by means of ball bearings about the brass tube that carries the guard cylinders; the lower end moves in an ordinary slide bearing around the steel pin under the lower guard cylinder. This lower end is provided with a pinion, so that the outer cylinder can be rotated by a tangent screw (fig. 3, *f*). The gearing rate is for cylinder I. 1 : 169 and for cylinder II. 1 : 171. On the lower end a ring-shaped dish is placed (fig. 3, *g*) containing phosphorus pentoxide for drying the air.

The Driving Mechanism.—In order to obtain a constant deviation the outer cylinder must rotate with constant speed. At first an electric gramophone motor with a centrifugal regulator was used. This gave a fairly constant speed if the tension were kept constant. The gramophone table was replaced by a pulley, transmitting the rotation to a smaller pulley at the tangent screw. In order to diminish and smooth out the small velocity variations a heavy balance wheel (80 kgr.) with ball bearings was coupled between the motor pulley and the tangent screw pulley. With this arrangement the earlier determinations already published * were carried out.

Even when using the balance wheel there occurred sometimes slow velocity variations as large as 0.2 per cent. These variations certainly do not cause an error of equal magnitude, but of course it is desirable to maintain as constant a velocity and deviation as possible.

* G. Kellström, 'Nature,' xxxvi. p. 682 (1935).

In continuing the determinations in the spring of 1936 the following arrangement was used: an electron valve-driven tuning-fork * produces an alternating current which is amplified and made to drive a small synchronous motor ("Saja" Kleinmotor from Schmidt & Haensch, Berlin). The latter was connected directly to the tangent screw by means of two pieces of rubber tubing and an aluminium bolt. The frequency of the tuning-fork (and the alternating current) depends partly on the temperature partly on the amplitude of the tuning-fork, which in its turn depends on filament current, anode- and grid potential of the driving valve. There is no difficulty in keeping these quantities sufficiently constant to maintain the frequency constant to one part in 10,000 during the time required for one determination (about 20 min.). By using different tuning-forks it was possible to get frequencies from 52 to 230, corresponding to times of revolution from 99 to 22 sec.

Determination of the Instrument Constant K.

(1) *The Measurements of a, l, and b.*—These measurements were made with precision micrometer calipers from C. E. Johansson, Eskilstuna, divided into hundredths of a millimetre, and allowing the thousandth place to be estimated. The screw errors are guaranteed to be less than 0.002 mm. The zero reading of the caliper was determined for each measurement by means of the accessory end gauges; for control a set of parallel end gauges, system C. E. Johansson, were used as well. All measurements were made at temperatures near 20° C., and the results reduced to 20°, using the following linear dilatation coefficients: for steel 1.15×10^{-5} ; for brass $1.85 \cdot 10^{-5}$; for electron metal 2.64×10^{-5} .

The diameter of the inner cylinder, $2a$, could be measured directly. The measurements were made at three different heights, and for every height at eight different places, symmetrically situated about the circumference. Such a series of measurements is reproduced in Table I., showing the accuracy and the exact cylinder shape.

* See A. Scheibe, *Jahrb. d. drahtl. Telegr. u. Teleph.* xxix. pp. 120, 158 (1927); W. Hensel, *Diss. Leipzig* (1928); H. H. Rust, *Diss. Leipzig* (1932). A similar arrangement has been used for viscosity measurements by R. K. Day, *Phys. Rev.* xl. p. 281 (1932).

The mean of all series made, reduced to 20°, is $2a = 5.63534$ cm.; $a = 2.81767$ cm., with an error probably less than 0.0001 cm. (The mean deviation from different series = 0.00004 cm.)

In the same way the height of the cylinder was obtained :

$$l = 9.9981 \pm 0.0005 \text{ cm.}$$

The measurement of b (the inner radius of the outer cylinder) was made through measuring first the outer diameter 20 mm. from the upper and lower end at eight or sixteen places on the circumference, and then the thickness of the cylinder wall at the same places, whereafter b is obtained as the difference between the outer radius and

TABLE I.

Diameters at Different Distances from Upper
End of Inner Cylinder.

	10 mm.	40 mm.	60 mm.	90 mm.
1.....	56.354	56.354	56.354	56.355
2.....	54	54	54	54
3.....	53	53	54	54
4.....	53	53	54	55
5.....	54	53	54	54
6.....	53	52	55	54
7.....	54	52	53	54
8.....	53	53	54	55
Mean ...	56.3535	56.3530	56.3540	56.3544
		56.3537 mm.		

the thickness. By special measurements it was proved that the outer diameter was the same at the middle of the cylinder as at the ends. In Table II. is given a series for cylinder I., and in Table III. one for cylinder II.

As is seen from these tables cylinder II. is considerably more carefully made than cylinder I. Most of the η -determinations were therefore made with cylinder II. The mean value of all measurements, reduced to 20°, were

For cylinder I. $b = 3.26628 \pm 0.0002$ cm.

„ „ II. $b = 3.18328 \pm 0.0001$ „

(2) *Determination of the Moment of Inertia I of the Suspended System.*—As it is impossible to calculate I

TABLE II.
Cylinder I. Temp. 20°.

No.	20 mm. from upper end.		20 mm. from lower end.	
	Diam.	Thickness.	Diam.	Thickness.
1	72.470	3.584	72.467	3.575
2	69	80	68	74
3	69	71	71	71
4	70	64	68	72
5	71	60	61	67
6	70	66	57	71
7	70	74	58	72
8	69	74	62	62
9	69	60	66	62
10	70	51	69	63
11	70	57	72	66
12	70	74	69	74
13	71	85	61	74
14	70	87	57	76
15	70	83	58	77
16	69	85	62	76
Mean	72.4698	3.5722	72.4641	3.5708
R =	36.2349	36.2321
b =	32.6627	32.6613

TABLE III.
Cylinder II. Temp. 21.3°.

No.	20 mm. from upper end.		20 mm. from lower end.	
	Diam.	Thickness.	Diam.	Thickness.
1	73.261	4.798	73.259	4.798
2	59	96	64	98
3	61	98	64	99
4	62	99	59	97
5	61	96	59	97
6	59	95	64	98
7	61	96	64	97
8	62	96	59	96
Mean	73.2608	4.7968	73.2615	4.7975
Temp. corr.	-7	-7
2 R ₂₀ =	73.2601	73.2608
2 R ₂₀ =	36.6301	36.6304
b ₂₀ =	31.8333	31.8329
Mean.....	31.8331

directly, attempts were first made to determine I by the well-known method of determining the period of vibration T of the cylinder alone, and then the period T_1 when a ring of known moment of inertia I_1 was added. The desired moment I is then calculated from the equation

$$I = \frac{T^2}{T_1^2 - T^2} \cdot I_1, \text{ assuming that the torsional couple con-}$$

stant of the suspension does not alter with the weight. It was found, however, that rings of different weights gave somewhat different results, owing to the fact that the torsional couple for the fine wires that must be used here decreases noticeably with the carried weight (the length of the wire increases, the diameter decreases, and the coefficient of rigidity decreases *), for which reason T comes out a little too large and I a little too small. This difficulty was overcome in the following way. A cylindrical disk is carefully made, with about the same weight and the same moment of inertia as the inner cylinder. In the centre it has a hole, fitting with friction to the central brass rod. The period of vibration is determined when the inner cylinder is put on the rod, and then when it is taken away and the disk put on instead. If the total moment of inertia about the axis of vibration is in the former case I with the period T , in the latter case

resp. I_1 and T_1 , $I = I_1 \cdot \frac{T^2}{T_1^2}$. I_1 is the sum of the moment of inertia of the disk, which can be accurately determinated, and of the central rod, which can be approximately calculated from the weight, 23.22 gr., and the radius 0.20 cm., and more accurately determined from vibration experiments.

Two similar disks were made (I. and II.), the first somewhat lighter, the second somewhat heavier than the inner cylinder, weighing 66.7 gr. Their outer diameters were measured with micrometer calipers at several points, as described above, and moreover in a comparator, where also the diameters of the central hole could be measured. The weights of the disks directly measured were corrected for the buoyancy of the air. The results of the measurements (at 20°) are given in Table IV.

The moment of inertia of the brass rod I^1 can be calculated to be 0.46 gr. cm.² from the data given. A small

* J. R. Benton, *Ann. der Phys.* iii. p. 471 (1900).

amount must be added to this because of the mirror and its holder. Through determinations of periods of vibration according to Table V. I^1 was found to be 0.49. Thus the moment of inertia of the system rod+disk I is 401.88 gr. cm.², and of rod+disk II. 432.81 gr. cm.².

In order to obtain the desired moment of inertia I a series of vibration periods were determined, when the inner cylinder was put on (T), and when it was replaced by disk I. (T_1) or II. (T_2). For the time determinations a chronograph with two electromagnetic pens was used, one of which was connected with the precision pendulum-clock of the institute, making one mark a second, and the other to a telegraph-key, placed near the telescope, where

TABLE IV.

	Disk I.	Disk II.
Outer radius	3.5003	3.5833
Inner „	0.2020	0.2024
Thickness	3.9468	0.9236
Weight.....	65.304	67.123
Moment of inertia	401.39	432.32

TABLE V.

T_1 .	T^1 .	I_1 .	I^1 .
31.870	1.1125	401.9	0.490
53.081	1.862	„	495
88.899	3.1012	„	489
Mean			0.491

the oscillations were observed. The periphery velocity of the chronograph drum was about 1 cm./sec., and the record of time could be determined to within 0.02 sec. After the vibrations had been started and going on for some time, so that the amplitude was sufficiently small, six or eight passages through the equilibrium positions were registered, and after an interval of 30 to 60 minutes another six or eight passages. The temperature was kept constant to 0.2°, and the vibrating system was effectively shielded from air currents. Each determination of I includes at least three period determinations, in the order $T_1 T T_1$ or $T T_1 T$. This was for the purpose of making certain nothing had been changed during the determination. Several different suspension wires were used. The determinations carried out in the spring of 1935 are

collected in Tables VI. and VII. The mean value is $I=423\cdot22$, with a mean error of about 0·05 per cent.

In connexion with the repeated η determinations in the spring of 1936 a new series of I-determinations was also made. The disks were then found to have increased their weight with a small amount, probably due to oxidation, while their dimensions were unchanged. Disk I. had increased 0·008 gr. (0·012 per cent.) to 65·312 gr.,

TABLE VI.

$$I_1=401\cdot88.$$

Susp. No.	T ₁	T.	I.
1	52·349	53·718	423·17
2	31·820	32·683	423·79
3	52·848	54·247	423·44
4	31·866	32·704	423·32
5	53·107	54·511	423·41
5	53·081	54·508	423·78
6	18·3834	18·8589	422·94
6	18·3865	18·8628	422·97
7	88·899	91·295	423·83
8	31·8795	32·7189	423·32
8	31·8648	32·7009	423·25
9	12·1364	12·4490	422·85
10	11·2656	11·5555	422·83
10	11·3021	11·5902	422·63
10	11·3042	11·5947	422·80
11	10·7229	10·9971	422·70
11	10·7221	11·0059	423·44
Mean		423·20	
Mean error....			0·32

Suspensions used in Tables VI. and VII. :—

- Nos. 1, 3, and 5 : phosphor bronze wire, 0·069 mm.
- Nos. 2, 4, and 8 : " " " 0·1 mm.
- No. 7 : phosphor bronze wire, 0·049 mm.
- No. 6 : chromnickel wire, 0·11 mm.
- No. 9 : manganin wire, 0·15 mm.
- No. 10 : molybdenum wire, 0·12 mm.
- No. 11 : tungsten wire, 0·12 mm.

and disk II. 0·014 gr. (0·02 per cent.) to 67·137 gr. Thus the new values of I_1 and I_2 become $I_1=401\cdot93$ and $I_2=432\cdot90$. The result of the new I-determinations is seen in Tables VIII. and IX. The small increase of the mean value $I=423\cdot28$ can partly be explained by the fact

that the weight of the cylinder has also increased (0.003 gr.).

(3) *The Value of the Instrument Constant K.*—After having determinated a , l , b , and I , K is computed from equation (3) p. 315. The variation of K with temperature is obtained by using the dilatation coefficients given on p. 319. The latest value of I gives

For outer cylinder I. $K=0.68210 [1-0.0000696(\theta-20)]$

„ „ „ II. $K=0.57728 [1-0.0000813(\theta-20)]$

TABLE VII.

$$I_2=432.81.$$

Susp. No.	T_2 .	T .	I .
5	55.100	54.498	423.40
5	55.076	54.478	3.46
6	19.0701	18.8593	3.29
6	19.0723	18.8622	3.33
7	92.252	91.265	3.60
8	33.0806	32.7189	3.40
8	33.0593	32.6999	3.45
9	12.5919	12.4500	3.13
9	12.5888	12.4467	3.09
10	11.6871	11.5559	3.14
10	11.6874	11.5554	3.09
10	11.7243	11.5902	2.97
10	11.7276	11.5947	3.06
11	11.1202	10.9971	3.28
11	11.1351	11.0059	2.82
Mean		423.23	
Mean error		0.18	

The Determination of η .

(1) *Adjustment of the Apparatus.*—The accurate centering of the guard cylinders and the outer cylinder about the common axis has been made certain through the method of construction. Check examinations have shown that the error in the centering does not exceed

0.016 mm. for the upper guard cylinder.

0.004 „ „ lower „ „

0.02 „ „ outer cylinder I.

0.004 „ „ „ „ II.

The inner cylinder, being vertical when suspended, is adjusted symmetrically between the guard cylinders by means of the horizontal adjustment screws at the upper end of the suspension and the levelling screws of the stand. Using a microscope with ocular scale the inner

TABLE VIII.

 $I_1 = 401.93$.

Susp. No.	T ₁ .	T.	I.
1	11.2748	11.5613	422.61
1	11.2902	11.5799	2.72
2	10.7541	11.0233	2.30
2	10.7500	11.0233	2.62
3	35.0173	35.9379	3.34
3	34.9709	35.9086	3.77
4	56.5708	58.0834	3.71
5	54.5273	55.9813	3.65
6	88.4805	90.8505	3.75
7	20.2208	20.7442	3.00
7	20.2164	20.7404	3.03
Mean			423.14
Mean error			0.46

TABLE IX.

 $I_2 = 432.90$.

Susp. No.	T ₂ .	T.	I.
1	11.7175	11.5805	422.79
2	11.1529	11.0227	2.85
2	11.1527	11.0228	2.88
3	36.3310	35.9356	3.53
3	36.3018	35.9095	3.60
4	58.6466	58.0293	3.84
5	56.5744	55.9907	4.01
6	91.8052	90.8339	3.79
7	20.9774	20.7449	3.36
7	20.9742	20.7440	3.45
Mean			423.41
Mean error			0.35

Suspensions used in Tables VIII. and IX:—

No. 1=molybdenum wire, 0.12 mm.

No. 2=tungsten wire, 0.12 mm.

No. 3=phosphor bronze wire, 0.1 mm.

Nos. 4 and 5="," 0.069 mm.

No. 6="," 0.049 mm.

No. 7=chromnickel wire, 0.10 mm.

cylinder can thus be brought to alignment with the guard cylinders (which in this way also are made vertical) within 0.02 mm. (see Pl. III. fig. 2). When this adjustment is made the outer cylinder has to be put on. By turning 1/4 revolution on the head screw of the suspension (fig. 3, *i*) the inner cylinder is made to rest on the lower guard cylinder. The whole system is then lifted from the stand, the outer cylinder is screwed on to the upper lid from below, the system is put back in the stand, and the inner cylinder is raised again with the head screw.

(2) *Determinations of the Angle of Deflexion ϕ .*—This determination was made by the usual mirror, telescope, and scale method. The mirror, from Steinheil Söhne, with diameter 15 mm. and thickness 0.33 mm., was of superior quality, as was also the telescope, so that readings to 0.1 mm. could be made. The scale, 1000 mm. in length, etched on thick plate glass, was compared with the invar standard metre of the institute. It was adjusted in a horizontal position with both ends equally distant from the mirror. The scale distance d , about 205 cm., was measured by means of a special measuring rod consisting of two silver steel rods, about a metre long, and a micrometer screw. By this arrangement distances from 2042 to 2055 mm. can be directly measured to 0.01 mm. To the directly measured scale distance is to be added a positive correction of 0.22 mm. for the thickness of the mirror, and a negative correction of 1.32 mm. for the glass window (thickness 3.97 mm.; index of refraction 1.50). At the adjustment of the inner cylinder it is arranged that the zero position (n_0) lies near the middle division (n) of the scale. If then the deflected position is indicated by n_1 , ϕ is calculated from the equation

$$\pm \phi = \frac{1}{2} \left(\operatorname{arc} \operatorname{tg} \frac{n_1 - n}{d} + \operatorname{arc} \operatorname{tg} \frac{n - n_0}{d} \right). \dots \quad (4)$$

(3) *Determination of t and T .*—For this purpose the chronograph already mentioned was used. When t is to be determined one pen is connected in series with a mercury contact (fig. 3, *h*), which is shut once for each revolution of the outer cylinder. This recording is continued for about 10 min., and during this time the deflexion is observed. To record T the chronograph pen is connected to the telegraph-key near the telescope, and a

number of passages in both directions through the zero position is recorded. Owing to the relatively large damping this recording can continue for only 10 to 12 min., which, however, permits the required accuracy. At the same time the turning points are read in order to calculate the logarithmic decrement and the damping correction, which amounts to 0.3 per cent. for the longest periods and 0.05 per cent. for the shortest period used. The damping is caused mainly by the 0.1 mm. air layer between the inner cylinder and the guard cylinders. Only a small part of the damping is caused by the air between the outer and inner cylinder.

The standard clock used for these records shows a lag not exceeding 0.5 sec. in 24 hours, and thus cannot cause any perceivable error in the time measurements.

(4) *The Temperature Control.*—As a temperature rise of 1° causes η to increase 0.27 per cent., it is necessary to determine the temperature with sufficient accuracy and to keep it constant during the experiment. The temperature was read on a mercury thermometer, divided in 0.1°, and compared with the standard thermometer of the institute. It was placed with the bulb near the middle of the outer cylinder and was read with a telescope to 0.01°. The whole apparatus was covered by a large zinc cylinder. During the first determinations no temperature regulation was used, and in exceptional cases therefore temperature changes as large as 0.3° might occur during one determination. During the later determinations, carried out in 1936, an automatic temperature control was procured, consisting of a Ramberg type mercury thermoregulator, with relay, electric heater, and two fans. This kept the temperature constant in the whole room. The variations within the plate cover were generally less than 0.03°. The value of η , determined at a temperature θ °, was reduced to 20°, using Millikan's (2) formula, stated to be valid in the interval 12°–30°.

$$\eta_{20} = \eta_{\theta} + 4.93 \cdot 10^{-7} (20 - \theta) \dots \dots \quad (5)$$

(5) *Description of a Typical η Determination.*—As an example of a complete η determination run no. 67 will be reproduced.

(a) The scale distance is measured. The reduced distance $d = 2047.6$ mm. (This distance was not measured

for every determination, but only when some adjustment had been made.)

(b) The inner cylinder is set into very small vibration by a small turn on the tangent screw, and the zero position determined by reading three successive turning points : 455.7, 455.9, 455.7. Thus $n_0=455.80$ at the beginning.

(c) The temperature is read : 20.06°.

(d) The outer cylinder is set rotating by starting the synchronous motor. By stopping it and starting it again at the right moment the inner cylinder can be brought to rest near the deflected equilibrium position. It first makes damped vibrations of the period T about this position, soon passing over to forced vibrations of the period t . The amplitude of these vibrations is always very small, in this case about 0.6 mm., and should be almost zero if the cylinders were perfectly cylindrical and coaxial. As this is not the case the couple acting on the inner cylinder is subject to small variations with the period t , even if the angular velocity of the outer cylinder is perfectly constant. The amplitude also depends on the ratio $t : T$.

As soon as constant conditions are attained a chronograph record of t is taken during 10 minutes, and simultaneously as many turning points as possible are read in the telescope (see Table X.). The mean of Table X., $n_1=14.70$, is corrected for the glass scale by subtracting 0.05 mm. (correction for scale division 10—0.04 mm. ; for division 20 —0.06 mm.). Thus the corrected deflexion is $n_1=14.65$.

(e) The temperature is read : 20.04°.

(f) The motor is stopped, and the inner cylinder is slowly brought back to its zero position by turning the tangent screw with decreasing velocity. Then the zero position is obtained from five successive turning points : 456.8, 454.9, 456.4, 455.2, 456.2, $n_0=455.76$. Mean of zero positions before and after the deflexion : 455.78. Correction for the glass scale : -0.02 ; corrected zero position : $n_0=455.76$. Thus

$$\phi = \frac{1}{2} \left(\operatorname{arc} \operatorname{tg} \frac{500-14.65}{2047.6} - \operatorname{arc} \operatorname{tg} \frac{500-455.76}{2047.6} \right) = 21775''.$$

(g) The inner cylinder is set vibrating by turning the tangent screw, the chronograph is connected to the telegraph-key, and T is determined by recording a number of passages through the zero position. At the same time

TABLE X.

Turning points.	
15.0	14.5
15.0	14.3
15.0	14.4
15.0	14.4
15.1	14.3
14.9	14.3
15.0	14.5
15.2	14.4
14.8	14.4
15.0	14.4
15.00	14.39

Mean : 14.70

TABLE XI.

Turning points.		Diff.
797.8		611.7
	186.1	480.4
666.5		376.9
	289.6	296.4
586.0		232.9
	353.1	183.2
536.3		144.1
	392.2	113.3
505.5		

the logarithmic decrement λ is determined by reading nine successive turning points (Table XI.), which gives $\lambda = 0.2092$.

(h) The temperature is read : 20.03° . As temperature of the determination no. 67 is taken the mean of the two first readings : $\theta = 20.05^\circ$.

(i) The time records are read. (The seconds are first numbered, starting from an arbitrary second mark before the first time record.) The result is given in Tables XII. and XIII.

The damping correction is

$$\Delta T = -\frac{(2 \cdot 303 \lambda)^2}{8\pi^2} \cdot T = -0 \cdot 376; \quad T_{\text{red.}} = 127 \cdot 644 \text{ sec.}$$

TABLE XII.

Determination of t .

No.	Reg.	No.	Reg.	6 t.
1	1.93	7	293.40	291.47
2	50.50	8	341.96	46
3	99.08	9	390.56	48
4	147.65	10	439.14	49
5	196.20	11	487.70	50
6	244.83	12	536.30	47
		Mean..	291.478	
		$t =$	48.580	

TABLE XIII.

Determination of T.

Passages in one direction.		Passages in the other direction.	
Reg.	3 T.	Reg.	3 T.
1.88		65.88	
129.87		193.90	
257.94		321.91	
385.90	384.02	449.95	384.07
513.90	.03	578.02	.12

$$\text{Mean : } 3 T = 384.060; \quad T = 128.020.$$

(j) According to the formula, p. 325, $K_{20.05} = 0.57728$.
Thus

$$\eta_{20.05} = 18171 \cdot 10^{-8}; \quad \eta_{20} = 18169 \cdot 10^{-8}.$$

The suspension used in this determination was a phosphor bronze strip from W. G. Pye & Co., Cambridge (no. 8256). This gave a fairly steady zero position. Phosphor bronze wires from Hartmann and Braun with diameters 0.049 and 0.069 mm. were also used. These gave periods of vibration of approximately 90 and 53 sec.

respectively. These wires showed some drift of the zero position at first, owing to the previous coiling, but after some time they could be used.

In order to eliminate errors caused by possible asymmetry of the apparatus (asymmetric adjustment of the scale etc.) the deflexions were taken in both directions (every second to the right and every second to the left).

The Result of the Determinations.

For the sake of completeness the results of the earlier determinations are given in Tables XIV. and XV.

TABLE XIV.

Outer Cylinder I.

Suspension.		Number of determ.	$\eta_{20} \cdot 10^8$ mean.	Mean error $\times 10^8$
No.	Description.			
1	Ph. br. wire 0.049	4	18198	38
2	" " "	7	18200	26
3	" " 0.069	6	18206	31
4	Ph. br. strip	8	18181	22
		Mean ..	18196	

TABLE XV.

Outer Cylinder II.

Suspension.		Number of determ.	$\eta_{20} \cdot 10^8$ mean	Mean error $\times 10^8$
No.	Description.			
5	Ph. br. wire 0.089	8	18217	7
6	" " 0.049	10	18206	13
7	" strip	8	18188	16
		Mean ..	18204	

The mean for cylinders I. and II. is $\eta_{20} = 18200 \cdot 10^{-8}$.

The results of the determinations made in 1936 with improved arrangements are given in Tables XVI.-XX., with a summary in Table XXI.

TABLE XVI.

Suspension 8. (Phosphor bronze strip.)

No.	<i>d.</i>	<i>n₀.</i>	<i>n₁.</i>	<i>t.</i>	T.	<i>θ.</i>	$\eta_{\theta} \cdot 10^8.$	$\eta_{20} \cdot 10^8.$
58..	2046.7	460.07	762.07	70.016	127.634	19.90	18220	18225
59..	"	460.08	157.67	70.005	127.618	19.59	18143	18163
60..	"	460.29	761.57	70.008	127.636	19.76	18174	18187
61..	"	460.17	157.09	69.995	127.638	19.80	18217	18226
62..	2047.1	459.51	751.91	72.301	127.648	20.40	18215	18195
63..	"	459.52	165.74	72.349	127.637	20.50	18213	18188
64..	"	450.54	742.33	72.406	127.667	20.76	18206	18169
65..	"	450.42	156.71	72.396	127.656	20.50	18200	18175
66..	2047.6	455.86	894.48	48.567	127.657	20.08	18226	18222
67..	"	455.76	14.65	48.580	127.644	20.05	18171	18169
68..	"	455.70	922.97	45.605	127.640	19.95	18207	18209
69..	"	455.72	20.62	49.220	127.640	20.03	18169	18168
70..	"	453.63	667.73	98.313	127.621	19.91	18186	18190
71..	"	453.58	238.35	98.285	127.638	19.96	18185	18185
76..	2047.3	456.83	0.88	47.064	127.645	20.10	18180	18175
77..	"	456.78	909.26	47.064	127.658	20.12	18207	18201
78..	"	457.31	232.61	94.124	127.644	20.02	18179	18178
79..	"	457.40	681.30	94.133	127.646	20.06	18199	18196
Mean							18190	

TABLE XVII.

Suspension 9. (Phosphor bronze wire, 0.049 mm.)

No.	<i>d.</i>	<i>n₀.</i>	<i>n₁.</i>	<i>t.</i>	T.	<i>θ.</i>	$\eta_{\theta} \cdot 10^8.$	$\eta_{20} \cdot 10^8.$
84..	2047.5	543.12	301.60	44.237	90.782	20.17	18229	18220
85..	"	543.89	786.46	44.241	90.792	20.15	18214	18206
86..	"	546.61	277.09	39.604	90.776	20.09	18206	18201
87..	"	547.01	818.48	39.617	90.770	20.12	18231	18225
88..	"	547.44	849.24	35.701	90.772	20.15	18233	18225
89..	"	547.24	247.80	35.716	90.773	20.17	18220	18211
90..	"	548.44	181.31	29.177	90.771	20.05	18201	18199
91..	"	549.06	919.93	29.179	90.773	20.17	18228	18219
92..	2047.6	489.61	939.44	23.997	90.765	20.00	18205	18205
93..	"	488.03	37.51	23.998	90.759	20.08	18193	18189
94..	"	488.98	858.70	29.073	90.769	20.15	18212	18204
95..	"	488.13	117.87	29.079	90.765	20.19	18207	18197
96..	"	488.00	282.50	51.923	90.764	20.09	18186	18181
97..	"	488.39	694.06	51.919	90.766	20.10	18203	18198
98..	"	488.95	975.66	22.256	90.764	20.19	18220	18210
99..	"	487.85	0.30	22.261	90.764	20.24	18207	18195
Mean							18205	

Discussion of the Accuracy.

The mean error (=the arithmetic mean of the deviations of the single values from the mean value) in Tables XVI.-XX. amounts to 10 units in the 5th figure of η , or about 0.06 per cent., and can be explained by the accidental

TABLE XVIII.

Suspension 10. (Phosphor bronze wire, 0.069 mm.)

No.	<i>d.</i>	<i>n</i> ₀ .	<i>n</i> ₁ .	<i>t.</i>	T.	$\theta.$	$\eta_{\theta} \cdot 10^8.$	$\eta_{20} \cdot 10^8.$
100..	2047.7	484.06	315.04	22.260	53.883	20.17	18212	18203
101..	"	484.83	653.61	22.261	"	20.24	18210	18198
102..	"	486.03	595.68	34.235	53.890	20.22	18209	18198
103..	"	485.93	376.20	34.242	"	20.23	18212	18200
104..	"	493.98	648.64	24.276	53.860	20.24	18213	18201
105..	"	493.80	338.97	24.281	"	20.27	18229	18215
106..	"	496.45	341.68	24.282	53.860	20.26	18224	18211
107..	"	496.80	651.50	24.281	"	20.29	18220	18205

Mean 18204

TABLE XIX.

Suspension 11. (Phosphor bronze strip.)

No.	<i>d.</i>	<i>n</i> ₀ .	<i>n</i> ₁ .	<i>t.</i>	T.	$\theta.$	$\eta_{\theta} \cdot 10^8.$	$\eta_{20} \cdot 10^8.$
108..	2046.0	486.33	0.58	43.932	127.257	20.36	18223	18205
109..	"	486.62	970.16	43.925	127.232	19.96	18203	18205
110..	"	486.75	838.22	59.931	127.242	20.05	18196	18193
111..	"	486.54	134.34	59.927	127.231	20.00	18195	18195

Mean 18200

TABLE XX.

Suspension 11. (Cylinder I.)

No.	<i>d.</i>	<i>n</i> ₀ .	<i>n</i> ₁ .	<i>t.</i>	T.	$\theta.$	$\eta_{\theta} \cdot 10^8.$	$\eta_{20} \cdot 10^8.$
112..	2046.0	505.35	807.15	59.205	127.296	20.15	18242	18234
113..	"	504.65	203.50	59.208	127.312	20.15	18213	18205
114..	"	502.18	926.30	42.339	127.300	20.06	18209	18206
115..	"	501.90	77.94	42.325	127.302	20.06	18203	18200
116..	"	501.03	998.54	36.268	127.304	20.05	18204	18201
117..	"	500.90	3.50	36.272	127.291	20.05	18210	18207
118..	"	500.20	745.38	72.530	127.294	20.01	18207	18207
119..	"	499.90	254.82	72.500	127.299	20.01	18191	18191

Mean 18206

TABLE XXI.

Summary of Tables XVI.-XX.

Table no.	Number of determina- tions.	$\eta_{20} \cdot 10^8$ mean.	Mean $\times 10^8.$
XVI.	18	18190	16
XVII.	16	18205	10
XVIII.	8	18204	5
XIX.	4	18200	6
XX.	8	18206	7

Mean 18201

errors of ϕ , t , T , and θ , according to the following estimation :—

A deviation of 0.2 mm. in $n_1 - n_0$ (if $n_1 - n_0 = 300$ mm.) changes ϕ and η by 0.067 per cent.

A deviation of 0.1 mm. in the scale distance d changes ϕ and η by 0.005 per cent.

A deviation of 0.01 sec. in t (if $t = 50$) changes t and η by 0.020 per cent.

A deviation of 0.01 sec. in T (if $T = 128$) changes T^2 and η by 0.016 per cent.

A deviation of 0.02° in θ changes η by 0.005 per cent.

Further, the instrument constant K enters in all determinations. The error in K is estimated as follows (valid for outer cylinder II.) :—

0.001 mm. in a makes an error of 0.033 per cent. in K .

0.001 „ „ b „ „ „ 0.022 „ „ „

0.005 „ „ l „ „ „ 0.005 „ „ „

0.05 p. c. „ I „ „ „ 0.05 „ „ „

Since the probability that all errors affect the result in the same direction is small, $\sqrt{\sum \Delta^2} = 0.064$ per cent. may be regarded as a probable limit of the error in K .

All errors hitherto mentioned should by the same method of calculation give a possible error in η of about 0.1 per cent., or

$$\eta_{20} = (18201 \pm 10) \cdot 10^{-8}.$$

Within these limits lie 39 of the 54 values in Tables XVI.—XX.

In addition to the above accidental errors several systematic errors may exist, out of which we may consider the following :—

(1) *The Influence of Slip.*—This effect has been investigated by Millikan *, Stacy †, States ‡, and Blankenstein §. At atmospheric pressure the effect is very small, and should for the apparatus here used give a positive correction of 0.002 per cent. for cylinder I. and

* R. A. Millikan, Phys. Rev. xxi. p. 217 (1923).

† L. J. Stacy, Phys. Rev. xxi. p. 239 (1923).

‡ M. N. States, Phys. Rev. xxi. p. 662 (1923).

§ E. Blankenstein, Phys. Rev. xxii. p. 582 (1923).

0.003 per cent. for cylinder II. It may therefore be neglected.

(2) *The Influence of Humidity on the Viscosity of Air* has been disputed, as the accuracy of the earlier determinations has not been sufficient to settle the matter. Thus Millikan * found that the viscosity of moist air is 2.2 per cent. *larger* than for dry air, Zemplén † found also an increase of 0.8 per cent., and Gilchrist ‡ an increase of 0.3 per cent.; while on the other hand Stearns § (with the same apparatus as Gilchrist) found that η *decreases* 0.3 per cent. when the air is saturated with moisture at atmospheric pressure. The last result is, from a theoretical point of view, more plausible, as the viscosity of water vapour is considerably less than that of air (although the simple mixing law is certainly not valid here).

TABLE XXII.

Determinations of η for Moist Air.
Suspension 8. Cylinder II.

No.	<i>d.</i>	n_0 .	n_1 .	<i>t.</i>	T.	θ .	$\eta_{\theta} \cdot 10^8$.	$\eta_{20} \cdot 10^8$.
72..	2047.6	437.99	867.97	49.292	127.632	20.00	18175	18175
73..	"	438.09	3.47	49.301	127.631	20.04	18143	18141
74..	"	437.93	651.25	98.581	127.670	20.04	18162	18160
75..	"	437.84	223.60	98.599	127.674	20.04	18128	18126
Mean								18151

Four determinations of η for moist air have also been carried out (nos. 72–75), the dish in the bottom of the outer cylinder then containing wet filtering paper instead of phosphorus pentoxide. The result may be gathered from Table XXII. The mean value is 0.22 per cent. less than the mean of the determinations for dry air, carried out with the same suspension (Table XVI.), in fairly good agreement with Stearns' result. (It is, however, possible that the air was not fully saturated during my determinations.)

(3) *Excentric Adjustment of the Inner Cylinder* in relation to the outer causes, according to a theoretic investigation

* R. A. Millikan, *Phil. Mag.* xix. p. 209 (1910).

† G. Zemplén, *Ann. der Phys.* xxix. p. 869 (1909).

‡ L. Gilchrist, *l. c.*

§ J. C. Stearns, *Phys. Rev.* xxvii. p. 116 (1926).

by Couette *, a larger value of the couple, thus a larger deviation and a larger value of η , than coaxial adjustment. Using cylinder II. an error of 0.02 mm. in the adjustment should, according to Couette, give 0.0074 per cent. too large a value, and an error of 0.1 mm. or 0.37 per cent. As the error probably does not exceed 0.02 mm., there was no reason to suspect any perceptible influence on the result. I have, however, made an experimental investigation of the matter. Four determinations of η were made with the inner cylinder excentrically adjusted in relation to the guard cylinders and the outer cylinder (det. nos. 80-83, Table XXIII.).

As is seen from Table XXIII. an evident decrease of the couple is obtained instead of the increase calculated

TABLE XXIII.

Inner Cylinder adjusted Excentrically.

No.	Excentricity.	<i>d.</i>	$n_0.$	$n_1.$	<i>t.</i>	T.	$\theta.$	$\eta_{\theta} \cdot 10^3.$	$\eta_{20} \cdot 10^3.$	Mean.
80..	0.38 mm.	2047.3	460.43	911.06	47.065	127.654	20.12	18133	18127	
81..	"	"	460.36	6.13	47.070	127.653	20.12	18122	18116	18122
82..	0.63 mm.	"	477.59	927.36	47.073	127.64	20.04	18074	18072	
83..	"	"	477.56	25.80	47.074	"	20.06	18067	18064	18068

by Couette. The decrease is—in relation to the mean value of Table XVI.,—for nos. 80-81, where the excentricity was 0.38 mm., 0.37 per cent., and for nos. 82-83, where the excentricity was 0.63 mm., 0.67 per cent., while according to Couette an increase of respectively 0.54 and 1.48 per cent. should have been expected. I have not as yet been able to find out the reason for this, as Couette's calculations would seem to be correct. However, it is of importance to state that all these three sources of error tend to give too low values of η .

(4) *Elastic Hysteresis of the Suspension.*—This effect, on the other hand, should give too high a value for the deflexion, and thus for η , since the zero position during the deflexion is displaced in the direction of the deflexion. An estimation of the amount of this effect can be obtained from the determinations of the zero positions before and after the deflexion. The mean value of this displace-

* M. Couette, *l. c.*

ment is found to be for suspensions 8 and 11 (phosphor bronze strips) 0.04 per cent. of the deflexion, for suspension 9 (0.049 mm. wire) 0.10 per cent., and for suspension 10 (0.069 mm. wire) 0.16 per cent. When the mean of both zero positions is taken the error should be half as large. It should be still smaller if only the zero position, taken immediately *after* the deflexion, is used. This method has therefore been applied for suspensions 9 and 10.

None of the systematic errors here discussed can probably cause an error amounting to 0.05 per cent. in η . If, however, this amount is added to that already calculated (0.1 per cent.) a limit of error of 0.15 per cent. is obtained, or

$$\eta_{20} = (18201 \pm 27) \cdot 10^{-8},$$

corresponding to

$$\eta_{23} = (18349 \pm 27) \cdot 10^{-8}.$$

If we use Millikan's data with this value of η in place of Harrington's value we would find for e

$$e = \left(\frac{18349}{18226} \right)^{3/2} \cdot 4.770 \cdot 10^{-10} = (4.818 \pm 0.011) \cdot 10^{-10} \text{ e.s.u.},$$

the uncertainty stated being due only to the viscosity.

From this result the following conclusions can be drawn :—

(1) The discrepancy between the oil-drop value and the X-ray value of e can very well be explained by the error in the η value used in calculating e by the former method. (New measurements by Bäcklin and Flemburg * seem to indicate that other errors may also exist.)

(2) For the present the X-ray value of e is to be considered the most accurate and probable.

My thanks are due to Professor M. Siegbahn for his many suggestions concerning the construction of the apparatus and for his generous help throughout the investigation.

* E. Bäcklin and H. Flemburg, 'Nature,' cxxxvii, p. 655 (1936).

XXXIV. The Intensity of the Characteristic Radiation as a Function of the Applied Voltage. By J. A. HILL and H. TERREY *.

IN connexion with some work on quantitative analysis by X-ray methods it was desirable to determine how the intensity of the charactersite radiation varied with the applied voltage especially with regard to the elements silver and copper. Several investigations dealing with a number of elements have been published (*cf.* references 1-9), but the results are not very conclusive. Two empirical equations have been proposed, viz.

$$(a) \quad I = K(V - V_K)^n,$$

$$(b) \quad I = K(V^2 - V_K^2),$$

where I is the intensity, K is a constant, V and V_K are the applied and critical voltages. The value of n lies between 1.5 and 2.0. In general, equation (a) has been obtained with radiation of long wave-length (1, 2, 3, 4, 5), and (b) with relatively short wave-length (6, 7, 8, 9).

In connexion with equation (b) Wooten pointed out that if I is plotted against V^2 the curve obtained is slightly bent at the beginning, but rapidly becomes a straight line. On extrapolation this straight line cuts the V^2 axis at a value slightly greater than V_K^2 . At high voltages Unnewehr (7) on making a similar plot found that a straight line was obtained for voltages up to about six times the critical value, after which the intensity appeared to approach asymptotically to a limiting value which he attributed to absorption of the radiation in the material of the anticathode.

Rosseland (10) has attempted to deduce an equation from theoretical considerations. He stated that

$$I = K \left[\frac{V - V_K}{V_K} - \log_e \frac{V}{V_K} \right]$$

is the upper limit for the intensity and that

$$I(V) = \frac{1}{KV} \frac{\delta I}{\delta V},$$

where $I(V)$ is the number of collisions of some inner group experienced per unit path by an electron excited by a

* Communicated by the Authors.

voltage V in which these electrons are actually ejected from the atom. When the voltage is high the filament stream of electrons will, of course, penetrate more deeply into the anticathode, but it seems quite feasible to suppose that the number of collisions per unit path will be approximately constant and thus independent of the velocity of the electron stream, and hence of the voltage.

If $I(V)$ is assumed to be constant, then

$$\frac{dI}{dV} = KV,$$

and thus $I = K \cdot (V^2 - V_K^2)$ which is the equation obtained in a number of cases.

If, however, $I = K(V - V_K)^n$, then

$$\frac{\delta I}{\delta V} = nK(V - V_K)^{n-1}.$$

$$\therefore I(V) = \frac{K'(V - V_K)^{n+1}}{V}$$

and $\frac{dI(V)}{dV} = K' \left[\frac{V(n-1)(V - V_K)^{n-2} - (V - V_K)^{n-1}}{V^2} \right].$

If $(V - V_K)$ be small then this differential coefficient is positive and large. If $\frac{dI(V)}{dV} = 0$, then

$$V(n-1) - (V - V_K) = 0,$$

i. e., $V = \frac{V_K}{2-n}$,

or $I(V)$ passes through a maximum if n is less than 2. It is extremely difficult to say what would be the form of the curve $I(V) = f(V)$, but it is hard to see any reason why it should exhibit a maximum.

Experimental.

In the present series of experiments the metal for examination was used as the anticathode of a Philips demountable X-ray tube. The applied voltage was measured by an electrostatic kilovoltmeter. In this connexion it should be emphasized that a fluctuating potential was used and the root-mean-square value was taken as the effective voltage. Thus it is obvious that any equation which assumes a constant voltage may not

be applicable under these circumstances. The radiations were photographed on Kodak films, and the degrees of blackening were compared with a Zeiss microphotometer.

As varying times of exposure were required depending upon the voltage used, a series of preliminary experiments was carried out in order to determine the Schwarzschild constant for the films. This was found to have a value of 0.9. It was also found that the degree of blackening was strictly proportional to the current in the tube.

Results.

In the case of copper the following results were obtained with variations of the voltage from the critical value 8.9 k.v. up to 48 k.v. :—

V (in k.v.)	8.9	14	17	23.5	30.5	36	42	48
I	0	1.0	1.85	3.99	7.81	10.99	15.70	20.05
$\log_{10}(V - V_K)$	$-\infty$	0.708	0.909	1.165	1.335	1.433	1.520	1.592
$\log_{10} I$	$-\infty$	0	6.267	0.601	0.893	1.041	1.196	1.302
$V^2 - V_K^2$	0	117	210	473	851	1217	1685	2285
$\frac{V - V_K}{V_K} - \log_e \frac{V}{V_K}$	0	0.120	0.263	0.670	1.196	1.648	2.168	2.207

Three graphs were plotted

$\log_{10} I$ against $\log_{10}(V - V_K)$ (fig. 1).

I .., $V^2 - V_K^2$ (fig. 2).

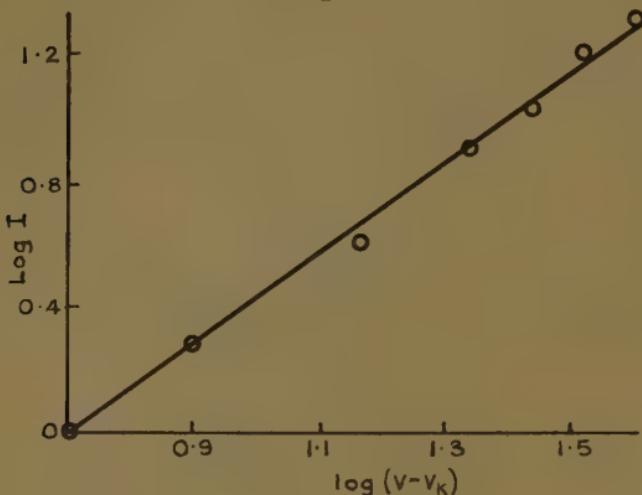
I .., $\frac{V - V_K}{V_K} - \log_e \frac{V}{V_K}$ (fig. 3).

From fig. 1 it is seen that $I = K(V - V_K)^{1.5}$ where K has a mean value of 0.08. The value of K was then calculated for each point separately and the mean departure from this mean was found to be 4.4 per cent. with a maximum deviation of 10.5 per cent. Similarly, from fig. 2, $I = 0.0089(V^2 - V_K^2)$; here the mean departure from the mean is only 2.6 per cent., whilst the maximum is 4.3 per cent. Fig. 3 shows a mean departure of 7.8 per cent. and a maximum departure of 18.6 per cent. It follows that the equation $I = K(V^2 - V_K^2)$ fits the results most satisfactorily.

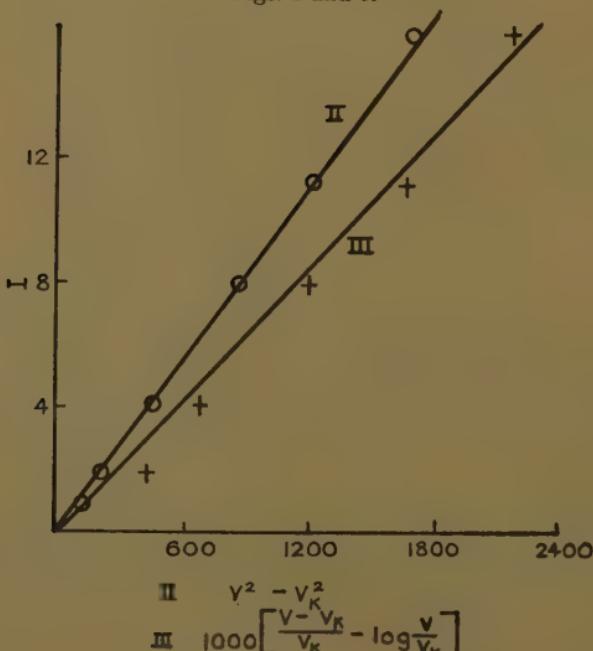
With silver the critical voltage is 25.5 and the range investigated was much more limited.

It is obvious that the last equation does not give results which are even of the right order. On plotting $\log I$

Fig. 1.



Figs. 2 and 3.

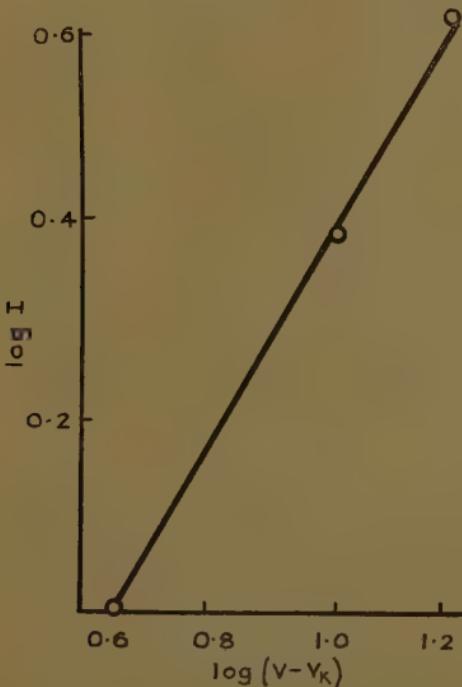


against $\log (V - V_K)$ (fig. 4) the gradient found was 1.2, *i.e.*, $I = K(V - V_K)^{1.2}$. Here K has a mean value of

0.155, with a mean departure from the mean of 4.1 per cent. and a maximum of 6.2 per cent. Fig. 5 shows that $I=0.00394(V^2-V_K^2)$ where the mean departure is 2.4 per cent. and the maximum 3.6 per cent. Thus for

V	25.5	30	36	41
I.....	0	1	2.45	4.14
$\log_{10}(V-V_K)$	$-\infty$	0.653	1.021	1.190
$\log_{10} I$	$-\infty$	0	0.389	0.617
$V^2-V_K^2$	0	250	646	1031
$\frac{V-V_K}{V_K} - \log_e \frac{V}{V_K}$	0	0.0139	0.0669	0.1328

Fig. 4.



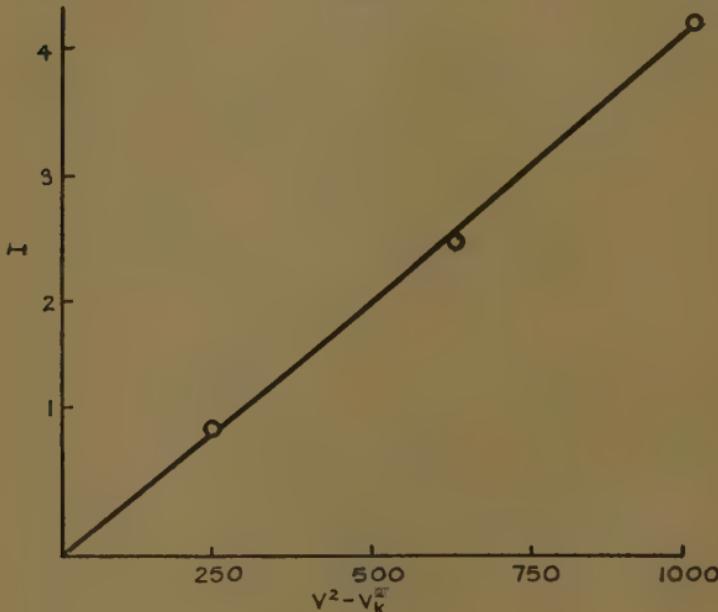
silver as with copper the best equation appears to be $I=K(V^2-V_K^2)$.

It will be noted that in the case of copper both $I=K_1(V^2-V_K^2)$ and $I=K_2(V-V_K)^{1.5}$ give results of the right order,

$$i. e., \quad K_1(V^2-V_K^2)=K_2(V-V_K)^{1.5} \text{ approx.}$$

This result is obtained if $\frac{V-3V_K}{2(V-V_K)^{1.5}}$ is very small. Hence if V be of the same order as $3V_K$ and $V-V_K$ is fairly large, then either of the equations will give results of the right order. In the case of silver, however, $V-3V_K$ is normally negative and $V-V_K$ much smaller than for copper and both equations cannot apply.

Fig. 5.



Summary.

The intensities of the K radiations from copper and silver have been investigated and it has been found that the equation $I=K(V^2-V_K^2)$ is the more correct form in these two cases.

References.

- (1) Webster and Clarke, *Phys. Rev.* ix. p. 571 (1917).
- (2) Stumper, *Z. Physik*, xxxvi. p. 1 (1926).
- (3) Jonsson, *Z. Physik*, xxxvi. p. 426 (1926).
- (4) Allison, *Phys. Rev.* xxx. p. 245 (1927).
- (5) Lorentz, *Z. Physik*, li. p. 71 (1928).
- (6) Wooten, *Phys. Rev.* xiii. p. 71 (1919).
- (7) Unnewehr, *Phys. Rev.* xxii. p. 529 (1923).
- (8) Kettmann, *Z. Physik*, xviii. p. 359 (1923).
- (9) Nasledow and Scharawsky, *Phys. Zeit.* xxviii. pp. 549 & 625 (1927); *Z. Physik*, xli. p. 155 (1927).
- (10) Rosseland, *Phil. Mag.* xlvi. p. 65 (1923).

XXXV. *Latent Energy in Combustion.*
*By Prof. W. T. DAVID *.*

IT has been suggested that during the passage of flame through an inflammable gaseous mixture there is formed in the flame front a very stable form of latent energy which remains in the flame gases left behind. It seems probable that the latent energy resides in metastable molecules of some kind or other which result from the reactions in the flame front.

This view has been evolved upon the basis of photographic studies of the intensity and duration of the after-glow in flame gases, continuous measurements of the temperature of thin platinum-rhodium wires immersed in these gases, and continuous pressure and heat loss measurements during explosion and subsequent cooling of inflammable gaseous mixtures in closed vessels.

Many hundreds of measurements have been made during the last seven years under widely varying conditions, and they show latent energies varying from 1 per cent. to 10 per cent. of the heat of combustion in large vessel explosions and from 5 per cent. to 28 per cent. of the heat of combustion in flames burning at constant pressure. Doubtless further experiments will show that these limits may be extended considerably in either direction.

It seems clear that the amount of latent energy expressed as a percentage of the heat of combustion (and, therefore, probably the proportion of the newly-formed molecules which are in the metastable condition) varies with the nature both of the combustible gas and of the diluent gases. It varies greatly also with the pressure at which combustion takes place, decreasing in amount as the pressure increases. It will be seen from the figures given above that it is much less in explosions than in flames, and this is satisfactorily accounted for if it be assumed that it is the pressure in the flame front that really matters, a view which finds support in the photographic studies of flame travel by H. B. Dixon, W. A. Bone, and others.

Ordinary gas calorimetry shows that the latent energy entirely disappears during the passage of flame gases through the calorimeter. The after-glow shows that if

* Communicated by the Author.

the latent energy is located in metastable molecules it tends to disappear during collisions when the flame gases are maintained in the homogeneous phase. The very long time during which the after-glow can be maintained (14 seconds, and probably much longer) suggests that the frequency of the required type of collisions, which after-glow intensity measurements show to be a function of the temperature, is very small even at high temperatures of the order of 2500°C . Consequently the rate at which the latent energy disappears in this way is very small. This is confirmed by the platinum thermometry and explosion pressure and heat loss measurements, which indicate that the latent energy remains practically undiminished in amount for a long time. It appears probable that the bulk of the latent energy disappears when the flame gases come into contact with surface.

The experimental evidence for the views expressed above is summarized in the following pages. Reference was made in a previous paper * to the peculiar behaviour of sodium atoms in flame gases.

Luminosity.

In closed vessel explosions the inflammable gases are completely inflamed by the time the maximum pressure is reached, but luminous radiation, which will be referred to as the after-glow, continues to be emitted after this. The spectrum of the after-glow differs from that of the luminous radiation from the flame front, except in the case of combustion of CO †.

The intensity of the after-glow has been shown to be dominated by the temperature of the flame gases ‡, that is to say, the gases left behind after the passage of the flame front, and if they can be maintained at high temperature (in the absence of surface §) the after-glow would last for a very long time indeed. Thus in an explosion made in a large explosion vessel in which the cooling of the flame gases took place slowly, the after-glow was seen

* Phil. Mag. xxiii. p. 251 (1937).

† Withrow and Rassweiler, Ind. and Eng. Chemistry, xxiii. p. 774 (1931).

‡ Phil. Mag. ix. p. 390 (1930).

§ An attempt was made many years ago to maintain luminosity in exploded gases contained in a small vessel kept at a high temperature in a furnace, but this was unsuccessful.

to exist for 14 seconds, and it is a fair inference from our experiments that, had a still larger explosion vessel been employed, so that cooling would have taken place even more slowly, the after-glow would have continued for a longer time than this.

Experiment has also shown that flame gases which have been cooled to darkness in the cylinder of a gas-engine by adiabatic expansion can be made luminous again by adiabatic compression *. This can be done over and over again with the same flame gases. Luminosity has been restored to these gases, even after they have been cooled to as low a temperature as 200° C., and there is no reason to suppose that luminosity could not have been restored after cooling to a much lower temperature had it been possible to arrange for an adiabatic compression which would have raised the gases to a temperature of the order of 1000° C.

The photographic studies of flame travel in tubes by H. B. Dixon, W. A. Bone, and others were not designed for the purpose of investigating the after-glow. They were made in narrow tubes in which cooling of the inflamed gases took place rapidly, and the after-glow was, therefore, of short duration. But even so the intensifying of the after-glow by means of compression waves can be most clearly seen. The long duration of the after-glow is manifest in the series of instantaneous snapshots by Ellis and Wheeler † taken during the cooling of exploded gases contained in transparent spherical and cylindrical vessels. Attention is specially directed to the sharp definition of the after-glowing portions of the gases.

Platinum Thermometry in Flames.

The platinum temperatures in flame gases were made during the "pre-pressure" interval in large vessel explosions. A thin platinum-rhodium wire (0.0005 inch diameter) was placed near the igniting spark. As the flame spreads outwards from the spark it met the wire and rapidly heated it up. The wire thereafter remained immersed in the flame gases and its temperature

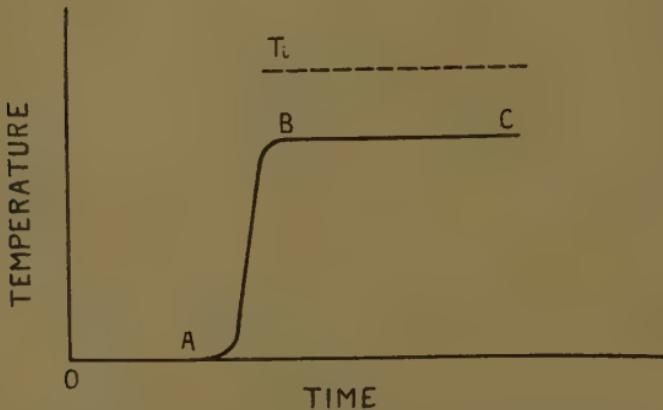
* Phil. Mag. ix. p. 396 (1930).

† Journ. Chem. Soc. Feb. 1927, p. 312. With much larger explosion vessels, in which cooling would have been slower, the duration of the after-glow would have been much longer.

(as inferred from its resistance) remained perfectly steady for as long as the pre-pressure interval lasted *.

A typical continuous record taken photographically of the wire temperature is shown in fig. 1. At 0 the spark is passed, at A the flame front reaches the platinum wire, and BC shows the steady temperature of the wire as it remains immersed in the flame gases during the pre-pressure interval. The ideal calculated temperature † which should be attained by the flame gases is shown by the dotted line T. The steady temperature BC is

Fig. 1.



* Except when very weak or very over-rich mixtures were inflamed. In such cases the wire after the rapid rise gradually decreased in temperature (see Phil. Mag. xii. p. 1043 (1931)). It has now become clear that in such mixtures chemical combination is not complete in the flame gases (see Bone, Newitt and Townend, 'Gaseous Combustion at High Pressure,' pp. 313 *et seq.* : Longmans, Green & Co., 1929), and the wire was heated up by surface combustion as well as by the hot gases. The surface combustion contribution to the wire temperature becomes less and less as combination in the homogeneous phase becomes more and more complete, and this accounts for the gradual decrease in its temperature. The attention is confined in this paper to medium and strong mixtures in which the wire temperature remained perfectly steady after its first rapid rise.

† The specific heat values used in this paper are those calculated by Johnston and Davies (J.A.C.S. lvi. pp. 271 and 1045 (1934)); Johnston and Walker (J.A.C.S. lvii. p. 682 (1935)); Kassel (J.A.C.S. lvi. p. 1941 (1934)); and Gordon (J. Chem. Phys. ii. pp. 65 and 549 (1934)). These values are just a little greater than those calculated by Nernst and Wohl and Johnston and Walker, which we proved, by means of high density large vessel explosion experiments to be "substantially accurate" (Phil. Mag. xviii. p. 307 (1934)), with the possible exception of the values for H_2O above about $2000^\circ C$. These, however, are not required for the present paper.

some hundreds of degrees C. below the calculated temperature T_i , and this cannot be accounted for to any appreciable extent by radiation from the flame gases or by radiation from the wire, for these have been shown to be small *. It can only be accounted for by assuming that a proportion of the original chemical energy of the inflammable gaseous mixture is held within the flame gases in some very stable and latent form.

Low as the platinum temperatures are in relation to the calculated temperatures, it seems likely that they are rather over-estimates of the true flame-gas temperatures (mean molecular translational energy) on account of the fact that the metastable molecules unload their energy upon the hot wire †. To some extent this is balanced by the small radiation losses from the flame gases and from the wire.

We are on safe ground in assuming that the wire temperatures give true flame temperatures for purposes of calculating the latent energy in the flame gases, in that the latent energy values so calculated are not likely to be over-estimates.

Our experiments included extended series of measurements with the following combustible gases:—Carbon monoxide, hydrogen, methane, acetylene, propane, ethylene, and propylene. The diluent gases in the case of the carbon monoxide mixtures consisted of argon, nitrogen, oxygen, carbon monoxide, and carbon dioxide. The

* *Phil. Mag.* xvii. p. 172 (1934), and xviii. p. 228 (1934).

† Many years ago we attempted to compare the mean temperature of exploded gases by means of platinum wires placed at various positions in the explosion vessel with that deduced from the explosion-pressure measurement (*Phil. Mag.* ix. p. 402 (1930)). The former was some hundreds of degrees above the latter. As the platinum wires were required to remain intact during the whole history of the explosion, we had necessarily to use weak mixtures in which, after inflammation, there was doubtless some incomplete combination. This probably accounts in some considerable measure for the large discrepancy between the wire temperature and the pressure temperature, and in this connexion it is of interest to recall that in wire temperature measurements of the flame during the pre-pressure interval in very weak hydrogen mixtures the wire temperatures exceeded even the ideal calculated temperatures by as much as 200° C. (*Phil. Mag.* xii. p. 1051 (1931)). Another contributing factor is the existence of a cold film of gas near the walls of the explosion vessel, which becomes of special importance when cylindrical explosion vessels are employed.

In the present paper attention is confined to medium and strong mixtures, and while in such cases a large excess of wire temperature over flame gas temperature is not to be expected, there are reasons for supposing that there is some excess.

pressures at which combustion took place was varied between 1 atmosphere and 5 atmospheres. Many experiments were also made at $\frac{1}{3}$ atmosphere, but some of them showed inconsistent results on repetition (probably on account of incomplete combination), and they have consequently been disregarded. Confining the attention to those mixtures in which consistent results were obtained on repetition of the experiments (sometimes frequent repetition, as, for example, in the case of check experiments), it is found that the defect of the steady platinum temperatures below the calculated temperatures ranged from 100° C. to nearly 600° C.* The latent energies calculated from the wire temperatures range from 5 per cent. to 28 per cent. of the heat of combustion. Some of the more important of these results have already been published † and it is not necessary to detail them here. They show that the amount of latent energy varies with

- (i.) the pressure at which combustion takes place ;
- (ii.) the nature of the combustible gas ; and
- (iii.) the nature of the diluent gases.

Explosion Pressures.

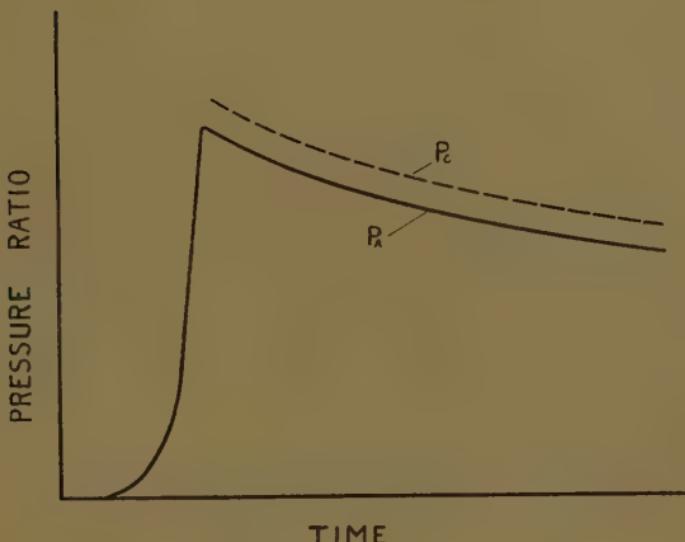
In these experiments inflammable gaseous mixtures were introduced into a spherical explosion vessel of 17.45 inches internal diameter and ignited by means of a spark-gap placed centrally. Continuous records of pressure and of heat loss were taken photographically during explosion and subsequent cooling. From the known heat of combustion of the mixture and the measured heat losses, the pressures which should exist at the moment of maximum pressure and during cooling can be readily calculated if it be assumed that the exploded gases are normal gases. The results of such calculations for a typical explosion are shown by the dotted curve P_c in fig. 2. The curve P_a in the same figure shows the actual continuous record of pressure obtained by means of a carefully calibrated optical diaphragm indicator.

* Dissociation is not taken into account in deriving the calculated temperatures. As dissociation is small at the measured platinum temperatures, a fairer perspective is obtained by comparing them with calculated temperatures in which dissociation is not taken into account.

† Phil. Mag. xxi. p. 280 (1936), and Proc. South Wales Inst. Engineers, p. 375 (1936).

It will be seen that the actual pressure falls below the calculated pressure by a remarkably steady amount during the initial stages of cooling *. This is clearly in line with the steadily maintained difference between the calculated temperatures and the platinum wire temperatures referred to in the previous section ; and again it would appear that the long continued sensibly constant defect between the calculated and the actual pressures can only be accounted for by assuming that a long-lived and stable latent energy remains in the exploded

Fig. 2.



gases. A precisely similar result is obtained from Hopkinson's well-known recording calorimeter experiment if recalculation is made upon the basis of the quantum specific heats †. This experiment, which permits of investigation much further down the cooling curve than do our experiments, further shows that the latent energy gradually disappears after the flame gases have cooled

* For simplicity of presentation attention is confined to mixtures in which dissociation is negligible ; but experiments with dissociating mixtures lead to precisely the same conclusion, namely, that the latent energy remains sensibly constant in amount during the initial stages of cooling.

† " Hopkinson's Explosion Experiments," 'The Engineer,' 3rd April, 1936.

to a mean temperature of 1000° C., and completely disappears by the time they have cooled to about 500° C. The time which the gases take to cool from 1000° C. to 500° C. is, however, relatively long, and it seems probable that during this time the flame gases are brought intimately into contact with the internal surface of the explosion vessel by convection currents. If this be so, it is important in connexion with the view already suggested that the latent energy disappears when the flame gases are brought into contact with surface.

The latent energy in the exploded gases corresponding to the defect between the calculated and the measured pressures is easily calculated, and some of our calculations have recently been published *. The latent energies so far found in large vessel explosions have ranged between 1 per cent. and 10 per cent. of the heat of combustion. These explosions were made with carbon monoxide, hydrogen, ethylene, and pentane as combustible gases, and with argon, nitrogen, oxygen, carbon monoxide, and carbon dioxide as diluent gases. The pressures of the inflammable mixtures before explosion were varied between $\frac{1}{4}$ atmosphere and 3 atmosphere. It was found—qualitatively in agreement with the platinum wire experiments—that the amount of latent energy varies with

- (i.) the initial pressure of the inflammable gaseous mixture ;
- (ii.) the nature of the combustible gas ; and
- (iii.) the nature of the diluent gases.

Correlation of Flame Temperature and Explosion Pressure Measurements.

It will be clear from the figures already given that the latent energies derived from flame temperature measurements during the pre-pressure interval are greater than those derived from pressure and heat loss measurements in explosions. As a typical example we will take the combustion of a 4.2 per cent. mixture of acetylene and air initially at 1 atmosphere pressure. From flame temperature measurements the latent energy is found to be 15 per cent. of the heat of combustion, whereas from explosion pressure and heat loss measurements it amounts only to 3.8 per cent. It has been stated that the latent

* Phil. Mag. xxii. p. 513 (1936).

energy decreases as the pressure of the inflammable mixture increases, and account should be taken of the fact that the average pressure at which the gases are burnt in the explosion is nearly four times that during the pre-pressure interval. But that is not enough to account for the discrepancy between the flame latent energy and the explosion latent energy, for experiment has shown that had the initial pressure been reduced to such an extent as to result in a mean pressure of 1 atmosphere during explosion the latent energy would still only have been about 5 per cent. of the heat of combustion.

Light is thrown upon this by an examination of the photographic studies of the travel of flame and its after-glow in long tubes containing inflammable gaseous mixture. These show that the after-glow becomes more intense with distance of flame travel, and particularly as the detonation phase is being reached. This indicates that the temperature of the flame gases left behind becomes higher, and the latent energy therefore becomes less *, with distance of travel of the flame front, and consequently with *pressure in the flame front*.

The suggestion that latent energy decreases with the pressure in the flame front provides an apparently completely satisfactory explanation as to why there is less latent energy in explosions than in flames. The latent energy is large at the centre as a result of combustion during the pre-pressure interval. As the flame travels further outwards it decreases, and near the walls it would be relatively small. In view of the fact that the gases in the central portion of the spherical vessel constitute only a small proportion of the gases in the spherical vessel, it will be clear that the latent energy averaged over the whole of the exploded gases will be small in comparison with that in the gases burnt during the pre-pressure interval †.

* This, of course, suggests that the number of metastable molecules become less ; but the influence of temperature is overwhelming.

It is suggested that the effect of turbulence resulting from skin friction and combustion in the surface layers should not be lost sight of considering the duration of the after-glow.

† The spectroscopic studies of Bone and his collaborators may be of interest in this connexion. They show that there is a marked foreshortening in the ultra-violet portion of explosion spectrograms as compared with flame spectrograms (Bone, Newitt and Townend, 'Gaseous Combustion at High Pressures,' p. 196 (Longmans, Green & Co., 1929) : and 'Nature,' cxxxiv, pp. 663 and 848.

It is therefore suggested as a more definitive statement that latent energy in flame gases is dependent upon

- (i.) the pressure in the flame front decreasing as the flame-front pressure increases ;
- (ii.) the nature of the combustible gas ; and
- (iii.) the nature of the diluent gases.

Disappearance of Latent Energy in Flame Gases.

In ordinary gas calorimetry the whole of the heat of combustion is accounted for. The stable latent energy must therefore disappear during the passage of the flame gases through the calorimeter. The after-glow shows that it tends to disappear when the temperature is sufficiently high to produce collisions of the kind which will enable the metastable molecules to attain the ground state *, but the not very great intensity of the after-glow and the very long time during which it may be maintained indicate that this must be a slow process. It seems probable that the bulk of the latent energy disappears when the flame gases come into intimate contact with surface, which of course they always do in gas calorimeters and in the various types of industrial plant. Indeed, owing to the turbulence set up as a result of skin friction and combustion in surface layers, it is very difficult to avoid intimate contact between flame gases and surface, and this is making it difficult to test our view by experiment. We have, however, three types of experiment in hand which have yielded promising preliminary results, and which it is hoped to complete in a few months. Meanwhile attention is directed to F. W. Lanchester's calorimeter experiment in which there was practically no surface, and in which he could only account for at most 85 per cent. of the heat of combustion of hydrogen †. Also, as indicating that flame gas metastable molecules can survive cooling in the absence of surface, it may be recalled that cooled flame gases in a large gas-engine cylinder were made luminous again when their temperature was raised by adiabatic compression. It seems

* It is possible that collisions in which free electrons take place are necessary. It is well known that flame gases are ionized, but ionization would not in and of itself account for the large latent energy.

† Proc. Inst. C.E. ccxxxvii. p. 200.

probable, too, that relevant mention may be made of the adiabatic compression ignition experiment of Tizard and Pye, in which explosion took place on the second compression *.

Summary.

Three types of experiment have been described, upon which the views expressed in the introductory section of this paper are founded.

XXXVI. *The Calculation of the Temperature Coefficient of a Magnet.* By J. R. ASHWORTH, *D.Sc.* †

IN a paper by the author in the 'Philosophical Magazine' for November 1915 on the application of Van der Waals's equation of state to magnetism ⁽¹⁾ a formula was derived for the temperature coefficient of a magnet. This formula was given in terms of the reduced intensity of magnetization (m) treated as a fraction of the maximum intensity (I_0) and the critical temperature (θ) in absolute degrees. Putting α as the temperature coefficient of magnetization, namely, $\frac{1}{I} \frac{dI}{dt}$, the equation given was

$$\alpha = \frac{8}{27} \frac{1}{\theta m(1-2m)} \dots \dots \dots \quad (1)$$

V. N. Thatte ⁽⁴⁾, in a paper published in May 1929 in this Magazine, derived a formula for the volume temperature coefficient of a liquid when the external pressure is negligible compared to the internal pressure, in which the probable value of one factor had to be found by reference to a series of experiments; a few months later Professor J. E. Verschaffelt ⁽⁵⁾, in a discussion of Thatte's paper, published a short investigation showing how Van der Waals's equation leads to a determination of the temperature coefficient of a liquid without having recourse

* D. R. Pye, 'The Internal Combustion Engine,' fig. 20, p. 98: Clarendon Press.

† Communicated by the Author.

to experiment for the evaluation of the factor mentioned above.

This development of Van der Waals's equation has its application to magnetism, and the equation given above for the temperature coefficient of a magnet may be rewritten in terms of reduced temperatures instead of reduced magnetic intensities and is applicable to any temperature up to the critical temperature.

If the external pressure be neglected Van der Waals's equation in terms of the density (ρ) of a fluid is

$$a\rho^2 \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) = RT, \dots \dots \dots \quad (2)$$

where a and R are constants and T is the absolute temperature.

Now it has been shown that an equation to ferromagnetism may be written analogous to Van der Waals's equation ⁽¹⁾, and so in the same way, when the externally applied field is zero or negligible compared to internal fields, we have

$$a'I^2 \left(\frac{1}{I} - \frac{1}{I_0} \right) = R'T, \dots \dots \dots \quad (3)$$

where I is the intensity of magnetization, corresponding to the density of a fluid, T is the absolute temperature, a' and R' are constants analogous to Van der Waals's a and to the gas constant R .

We proceed as follows: the temperature coefficient of a magnet is

$$\alpha = \frac{1}{I} \frac{dI}{dt} \dots \dots \dots \quad (4)$$

Differentiating equation (3) we have

$$a' \frac{dI}{dt} - 2 \frac{a'I}{I_0} \frac{dI}{dt} = R', \dots \dots \dots \quad (5)$$

which gives

$$\alpha = \frac{1}{I} \frac{dI}{dt} = \frac{R'}{a'I \left(1 - \frac{2I}{I_0} \right)},$$

or, more conveniently,

$$\frac{1}{\alpha} = \frac{a'I - 2 \frac{a'I^2}{I_0}}{R'} \dots \dots \dots \quad (6)$$

From (3) we have

$$2\alpha'I - 2\alpha' \frac{I^2}{I_0} = 2R'T$$

or $\alpha'I - 2\alpha' \frac{I^2}{I_0} = 2R'T - \alpha'I,$

and therefore

$$\frac{\alpha'I - 2\alpha' \frac{I^2}{I_0}}{R'} = 2T - \frac{\alpha'I}{R'}.$$

Hence from (6)

$$\frac{1}{\alpha} = 2T - \frac{\alpha'I}{R'} \dots \dots \dots \quad (7)$$

When the temperature is low I will be nearly the same as I_0 , the maximum intensity, and from the equation to the critical temperature (θ), namely,

$$\theta = \frac{8}{27} \frac{\alpha'I_0}{R'} \dots \dots \dots \quad (8)$$

we get, by substitution,

$$\frac{1}{\alpha} = 2T - \frac{27}{8} \theta \dots \dots \dots \quad (9)$$

or $\alpha = \frac{1}{\theta \left(2 \frac{T}{\theta} - \frac{27}{8} \right)} \dots \dots \dots \quad (10)$

Since the negative term in the bracket is always the larger the temperature coefficient will be decremental.

The equation is only strictly applicable at low temperatures when it is allowable to treat I as equivalent to I_0 .

Examples which are given in a table below show that the calculated values of α by this formula are of the right order and approximately of the right magnitude.

If it is not legitimate to treat I as equivalent to I_0 a further development is necessary. Equation (7) yields

$$\frac{1}{\alpha} = \theta \left(2 \frac{T}{\theta} - \frac{\alpha'I_0}{R'\theta} \frac{I}{I_0} \right) \dots \dots \dots \quad (11)$$

or $\frac{1}{\alpha} = \theta \left(2n - \frac{27}{8} m \right), \dots \dots \dots \quad (12)$

where n is written for the reduced temperature, (T/θ) , and m for reduced intensity of magnetization, treated as a fraction of the maximum intensity (I/I_0) .

The reduced ferromagnetic equation when the external field is zero or negligible is ⁽²⁾

$$m - m^2 = \frac{8}{27} n. \quad \dots \dots \dots \quad (13)$$

Taking the greatest root of this quadratic it becomes, on development,

$$m = 1 - \frac{8}{27} n - \left(\frac{8}{27} n \right)^2 - 2 \left(\frac{8}{27} n \right)^3 - \dots, \dots \quad (14)$$

and substituting this value of m in equation (12) we get

$$\frac{1}{\alpha} = \theta \left(2n - \frac{27}{8} + n + \frac{8}{27} n^2 \right), \quad \dots \dots \quad (15)$$

neglecting n^3 and higher powers of n .

Therefore

$$\alpha = \frac{1}{\theta \left(3 \frac{T}{\theta} + \frac{8}{27} \frac{T^2}{\theta^2} - \frac{27}{8} \right)}. \quad \dots \dots \quad (16)$$

Equation (16) has been used in calculating the temperature coefficients at the higher temperatures given in the table below. It will be seen that the concordance between calculated values and those experimentally found is satisfactory. It should be mentioned that the experiments to determine α were confined to the pure ferromagnetic metals, iron, nickel, and cobalt in the annealed state and in the absence of an external field, the magnets being of large dimension ratio so that self demagnetization was negligible; the temperature coefficient is then characteristic of the ferromagnetic metal.

When $T=0$, that is at absolute zero, the intensity then being a maximum, the formulas lead to

$$\alpha\theta = -\frac{8}{27}, \quad \dots \dots \dots \quad (17)$$

which shows that the temperature coefficient is inversely proportional to θ . More generally for any given reduced temperature n the characteristic temperature coefficients of the pure ferromagnetic metals are respectively in-

versely as their critical temperatures. This has been verified in a former investigation (3).

By means of the foregoing formula for $\alpha\theta$ an experimental determination of α , at a low temperature, would allow θ to be estimated, and this may sometimes be useful where a determination of the critical temperature is inconvenient to make or, if made, is in doubt.

It is useful also to note that

$$\alpha\theta = \frac{1}{m} \frac{dm}{dn}, \quad \dots \quad (18)$$

a formula in which only reduced quantities appear.

The equations and calculations given above have been made on the assumption that the critical temperature may be written

$$\theta = \frac{8}{27} \frac{a}{bR}$$

for fluids, following Prof. Verschaffelt, but the objection may be urged that when the external pressure is zero or negligible the equation to the critical temperature should be

$$\theta_{p=0} = \frac{1}{4} \frac{a}{bR} \quad \text{for fluids,}$$

$$\text{and} \quad \theta_{H=0} = \frac{1}{4} \frac{a'T_0}{R'} \quad \text{for magnetism.}$$

If this contention is sustained then the factor 8/27 and its reciprocal is to be replaced in the equations by 1/4 and its reciprocal respectively.

Thus the abbreviated formula for the temperature coefficient becomes

$$\alpha = \frac{1}{\theta \left(\frac{2T}{\theta} - 4 \right)}, \quad \dots \quad (19)$$

which is applicable for low temperatures.

The results of this formula are given in the table by starred figures, and now the agreement with experiment can hardly be improved.

But calculations carried out by the longer formula for higher temperatures in which the factor 8/27 is

replaced by $1/4$ yield results which deviate from experiment more widely than those in the table.

In the foregoing investigation the ferromagnetic equation has been written after the manner of Van der Waals's equation of state to a fluid, as this equation expresses in the simplest form the two salient facts of ferromagnetism, namely, the mutual action of the magnetic molecules and the existence of a limit to the intensity of magnetization. But any equation of state which recognizes successfully these conditions might have been used. The evidence for this is, as has been shown, that water, as a typical

Range of temperature in C°.	Mean T/θ .	α .	
		Calc.	Found.
(a) { 21-76	.484	0.00063	0.00052
		*0.00050 } {	
(b) { 50-107	.527	0.00065	0.00053
		*0.00052 } {	
(a) { 251-310	.835	0.0023	0.0024
(b) { 280-341	.882	0.0031	0.0030
(a) 237-287	.503	0.00040	0.00032
		*0.00032 } {	
(b) 650-700	.896	0.0021	0.0021
(a) 209-300	.388	0.00028	0.00024
		*0.00023 } {	
(b) 645-747	.712	0.00067	0.00064

(a) Calculated by equation (10) and starred figures by equation (19).

(b) " " (16).

liquid, and nickel, as a representative ferromagnetic, under variations of temperature up to their critical points, which are nearly the same, behave almost identically⁽³⁾. Thus an equation of state which applies to one will be equally good for the other.

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- (1) Ashworth, Phil. Mag. xxx. (November 1915).
- (2) Ashworth, Phil. Mag. xxxiii. (April 1917).
- (3) Ashworth, Phil. Mag. xxiii. (January 1912).
- (4) Thatte, Phil. Mag. ser. 7, vii. (May 1929).
- (5) Verschaffelt, Phil. Mag. ser. 7, viii. (December 1929).

XXXVII. *A Dynamical Method for the Measurement of Young's Modulus for Imperfectly Elastic Metals, and the Application of the Method to Nickel and some of its Alloys. By R. M. DAVIES, M.Sc., F.Inst.P. and I. H. THOMAS, B.Sc., Ph.D., University College of Wales, Aberystwyth **.

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I. INTRODUCTION.

THE aim of the present work is to determine the value of Young's Modulus at room-temperature for certain annealed and unannealed specimens of nickel and nickel alloys; the resistivities and temperature variation of resistance of these particular specimens have already been investigated and the results given in a previous communication ⁽¹⁾. The specimens are in the form of rectangular strips, about 0.6 cm. wide and 0.05 cm. thick; their composition and heat-treatment have been fully described in the communication referred to.

The methods available for the determination of Young's Modulus fall naturally into two classes—statical and dynamical methods; a discussion given in previous papers ⁽²⁾ † shows that on general grounds dynamical

* Communicated by Vice-Principal Gwilym Owen, M.A., D.Sc.

† Hereafter these papers will be quoted as "D. & J."

methods are superior to statical methods, both as regards accuracy and convenience. In the present work, it will be shown in the next Sections that a dynamical method possesses further advantages in the case of substances such as annealed substances, which are "soft" or imperfectly elastic, and also in the case of ferromagnetic substances. It was therefore decided to carry out the measurement by a dynamical method and to adopt the method described in D. & J.

In this method, Young's Modulus, E , is calculated from the natural frequency n_0 (or the natural pulsatance $\omega_0 = 2\pi n_0$) of transverse vibration of a uniform bar of the material vibrating in the fundamental mode; the bar is clamped at one end and is loaded at the free end with a stalloy armature of mass M . If ρ be the density of the bar, l its vibrating length, and A its cross-sectional area, then the mass M' of the vibrating bar is equal to $\rho A l$; the ratio of the mass of the load to that of the bar is $M/M' = c$ (say). If k is the radius of gyration of the cross-section of the bar about an axis through its centre of mass, perpendicular to the plane of motion, then E is given by the equation ⁽³⁾ :—

$$E = \frac{l^4 \omega_0^2 \rho}{k^2 z^4}, \quad \dots \dots \dots \quad (1.1)$$

where the parameter z is defined by the equation ⁽³⁾.

$$(1 + \cosh z \cos z) = cz (\cosh z \sin z - \sinh z \cos z). \quad (1.2)$$

The value of z is thus determined by c , the ratio of the mass of the load to that of the bar; the value of z corresponding to a given value of c can be determined by solving equation (1.2), but, in practice, it is more convenient to use the following explicit algebraic solution ⁽⁴⁾ :

$$z^4 = \frac{210}{8c+1} \left\{ \left[4c+1 \right] - \left[16c^2 + \frac{31}{35}(8c+1) \right]^{\frac{1}{2}} \right\}. \quad (1.3)$$

The error committed in evaluating z from this equation does not exceed 1 in 8000.

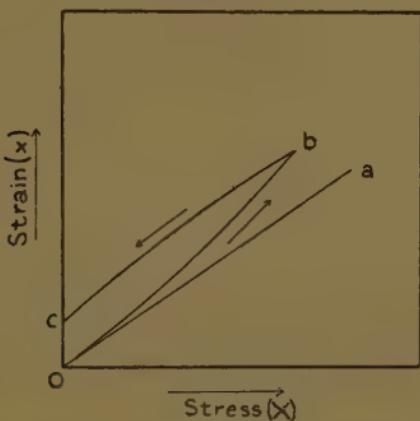
It is thus seen that in order to evaluate E , it is necessary to determine ρ , l , k , c , and ω_0 . The determination of the first four quantities presents no great difficulty. The evaluation of ω_0 is carried out by subjecting the stalloy armature fixed to the free end of the bar to the action

of an alternating magnetic field superposed on a steady polarizing field ; this results in a forced vibration of the bar with a pulsatance equal to that of the alternating driving field. The apparent resonant pulsatance ω_r of the bar under these conditions is determined as described in D. & J. (v. also Section IV. below). The apparent resonant pulsatance ω_r differs from the true natural pulsatance on account of the interaction between the stalloy armature and the magnetic fields ; it is, however, possible to derive ω_0 from ω_r by extrapolation to the limiting conditions of zero driving field and zero polarizing field.

II. THE DEFINITION AND MEASUREMENT OF YOUNG'S MODULUS OF IMPERFECTLY ELASTIC SUBSTANCES.

As stated in the introductory section, the annealed specimens used in the present work tend to be "soft"

Fig. 1.



Strain-stress curves for elastic materials (Oa), and imperfectly elastic materials (Obc).

or imperfectly elastic, and they acquire a permanent set when subjected to small stresses. It follows that they do not obey Hooke's Law for finite deformations, *i. e.*, for such deformations, the stress-strain relation is not linear and hysteresis effects are present.

This behaviour may be illustrated by reference to fig. 1, in which stress, X , is plotted as abscissa and strain, x , as ordinate. On such a diagram, the stress-strain curve

for a substance obeying Hooke's Law is a straight line, such as Oa , whilst the curve for a substance which does not obey Hooke's Law and which shows permanent set will be of the form Ob for stress increasing and of the form bc for stress decreasing; Oc represents the permanent set acquired as a result of the maximum stress applied.

Now, defining Young's Modulus in the usual way as the ratio of an increment of stress to the corresponding increment of strain, it is clear that the value of Young's Modulus, E , appropriate to any point on Oa , Ob , or bc , is given by the equation

$$\frac{1}{E} = \frac{\partial x}{\partial X}, \quad \dots \quad (2.1)$$

and $1/E$ is therefore equal to the slope of the stress-strain curve. For substances which obey Hooke's Law, $1/E$ and therefore E , are independent of X , but for substances which do not obey this law, the value of E is a function of the stress X on account of the curvature of the stress-strain curve. In addition, the value of E is a function of the previous history of the specimen because of the phenomenon of permanent set and hysteresis.

For these reasons, in the case of a substance which does not obey Hooke's Law and which shows permanent set, it appears best to define Young's Modulus as the reciprocal of the slope of the strain-stress curve at the origin, using a specimen which has not been subjected to a stress sufficient to cause a permanent set. If the value of Young's Modulus defined in this manner be called E_0 , then

$$\frac{1}{E_0} = \left(\frac{\partial x}{\partial X} \right)_{X=0} \quad \dots \quad (2.2)$$

If this definition be accepted, then static methods are inherently at a disadvantage, since it is difficult to measure accurately the small strains necessary to find the slope of the strain-stress curve at the origin.

It is possible, nevertheless, to adapt the dynamical method to the determination of Young's Modulus of this type of material without much loss of accuracy; this is possible because the quantity measured in a

dynamical method is a frequency rather than a displacement.

Near the strain axis (Ox) of fig. 1, the relation between stress and strain may be expressed as a power series of the form

$$x = a_0 + a_1 X + a_2 X^2 + \dots \dots \dots \quad (2.3)$$

where $a_0, a_1, a_2 \dots$ are constants.

Hence,

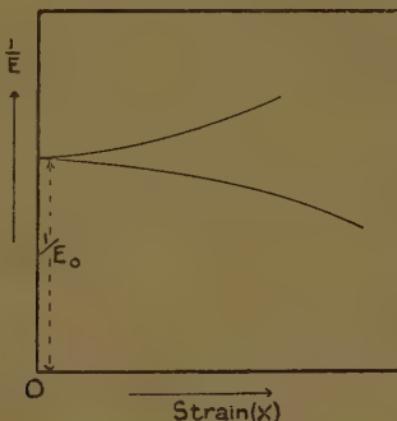
$$\frac{\partial x}{\partial X} = a_1 + 2a_2 X, \dots \dots$$

or, if we express $(\partial x / \partial X)$ as a function of x ,

$$-\frac{\partial x}{\partial X} = b_1 + 2b_2 x + \dots \dots \dots \quad (2.4)$$

where b_1, b_2, \dots are constants.

Fig. 2.



The relation between the reciprocal of Young's Modulus and strain for imperfectly elastic materials.

Thus, if E be the value of Young's Modulus appropriate to a stress X and a strain x ,

$$\frac{1}{E} = b_1 + 2b_2 x + \dots \dots$$

and $1/E_0 = b_1$ from equation (2.2).

Hence,

$$\frac{1}{E_0} = \frac{1}{E} - 2b_2 x \dots \dots \dots \dots \dots \quad (2.5)$$

Thus, if $1/E$ be plotted against x , the curve obtained will be similar to one of the curves shown in fig. 2, the

curve being concave upwards or downwards according to the sign of b_2 . It is clear that the intercept of the curve of the $(1/E)$ axis gives $1/E_0$ whatever be the sign of b_2 and it is this fact which is utilized for the determination of E_0 .

In the present experimental arrangement, the strain corresponds to the amplitude of vibration, and extrapolation to conditions of zero strain thus implies extrapolation to zero amplitude, *i. e.*, to the limiting conditions of zero driving field and zero polarizing field. In this respect, the extrapolation for imperfectly elastic substances is exactly the same as that described in D. & J. for substances which obey Hooke's Law.

It remains to obtain a quantity which can be determined by experiment and which is proportional to $1/E$. It is seen from equation (1.1) that, for a bar of fixed length loaded with a load of fixed mass, E is proportional to ω^2 . Thus, $1/\omega_r^2$ may be plotted instead of $1/E$ for the purpose of extrapolating to zero strain and the value of E_0 may be calculated from equation (1.1) provided that we use ω_0 , the value of ω_r corresponding to zero strain.

Experiment shows that the dynamical method, when applied in this manner, does give consistent results for this type of substance (*v. Sections V. and VI.*).

III. EXTENSION TO THE CASE OF IMPERFECTLY ELASTIC. FERROMAGNETIC SUBSTANCES.

Some of the alloys examined, in addition to being imperfectly elastic, were ferromagnetic in character. The Young's Modulus of ferromagnetic substances is known to be a function of the magnetic field in which the material is situated; this was shown to be the case by Honda and Tanaka ⁽⁵⁾, who carried out a systematic investigation on the binary alloy systems Fe-Co, Fe-Ni, and Fe-C. Their results have been extended by Nakamura ⁽⁶⁾ in an exhaustive investigation of the effect of magnetic field on the Young's Modulus of the Fe-Ni alloy systems.

It follows that a determination of the Young's Modulus of ferromagnetic substances should specify the magnetic field in which the specimen is placed during the determination. In the case of the present experiments, it is

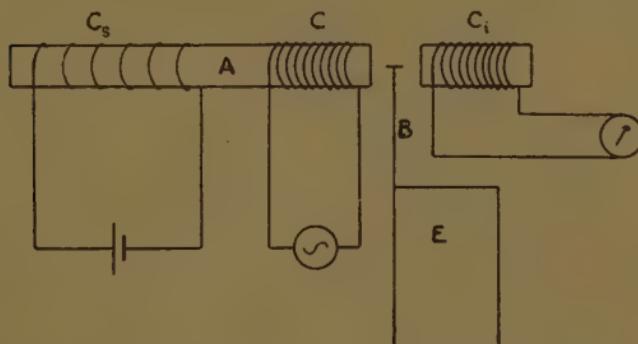
clear that the extrapolation to limiting conditions (as described in Section II.) will lead to the value of Young's Modulus for zero magnetic field. The results given in Section VI. show that the dynamical method, when applied in this manner, does lead to consistent results for ferromagnetic, imperfectly elastic substances.

IV. METHODS FOR DETERMINING THE APPARENT RESONANT PULSATANCE.

(a) *Existing Methods.*

Two methods for determining the apparent resonant pulsation ω_r of the bar have been described in D. & J. The first method consists in measuring the pulsatance at which the amplitude of vibration of the bar is a

Fig. 3.



Diagrammatic representation of "free" e.m.f. and current method.

maximum. The second method consists in measuring the pulsatance at which the "motional impedance" of the driving coil (*i. e.*, the coil carrying the driving current) is a maximum.

(b) "Free" E.M.F. and Current Method.

Another method, which may be called the "free" e.m.f. or "free" current method has now been investigated and has been shown to be capable of giving an approximate value of ω_r . This method may be described by reference to fig. 3, in which A represents the soft-iron core on which are wound C_s and C, the magnetizing and driving coils, carrying the polarizing and driving currents respectively.

B represents the bar of the material to be investigated, clamped at its lower end to the block E and carrying a small stallyoy armature at its upper end. A third coil, C_i , is placed to the right of the bar B, and the e.m.f. or current in C_i can be measured by an alternating current voltmeter or ammeter. The coil C_i is supported by a non-conducting, non-magnetic core, and thus there are no eddy-current or hysteresis effects in this coil.

Let i be the instantaneous value of the driving current in the coil C, I its virtual value, and ω its pulsatance.

With the bar B at rest, let M be the mutual inductance between the coils C and C_i ; under these conditions, let e_d and i_d be the instantaneous values of the e.m.f. and current in C_i , and E_d and I_d the virtual values; these values may be termed the "damped" values of the e.m.f. and current respectively. Let R_i , L_i , and Z_i be the resistance, inductance, and impedance of C_i when the bar is damped, *i.e.*, the "damped" values of these quantities.

Then

$$\left. \begin{aligned} e_d &= -j\omega M i; \quad E_d = \omega M I; \quad j = \sqrt{-1}, \\ i_d &= \frac{-j\omega M i}{R + j\omega L_i}; \quad I_d = \frac{\omega M I}{[R_i^2 + \omega^2 L_i^2]^{\frac{1}{2}}} = \frac{\omega M I}{Z_i}. \end{aligned} \right\} \quad (4.1)$$

When the bar is allowed to vibrate, then, in addition to the "damped" e.m.f. and current in C_i , there will be an e.m.f. and current due to the motion of B. Let the driving current I be kept constant and let the instantaneous value of the e.m.f. due to the motion of the bar be e_m and the instantaneous value of the current due to the motion of the bar to i_m ; these quantities are the "motional induced e.m.f." and the "motional induced current" in the coil C_i . The values of e_m and i_m will clearly depend on the amplitude of vibration of the bar.

If e_f be the instantaneous value of the e.m.f. in C_i when the bar is free, then e_f may be found by the vector addition of e_d and e_m ; e_f may be termed the "free" e.m.f. in the coil C_i . Similarly, the instantaneous value of the "free" current i_f in C_i will be obtained by the vector addition of i_d and i_m . Let E_f and I_f be the virtual values of the "free" e.m.f. and current respectively. If E_f or I_f be plotted against ω , it is clear that some change in the form of the curves should be found in the neighbourhood of the resonant pulsatance ω_r .

The theoretical and experimental investigation of the variation of E_f and I_f with ω is given in the following sections: in section (c) is given the theoretical investigation for the simplest case—that in which the effects of eddy-currents and hysteresis in the core A can be neglected. In section (d), the theory is extended so as to include the effects of eddy-currents and hysteresis in the core A, whilst section (e) contains experimental results.

In all cases, it will be assumed that the amplitude of vibration is small.

(c) *Theory of the "Free" E.M.F. and Current Method, neglecting Eddy-Current and Hysteresis Effects.*

Let m , s , and r be respectively the equivalent mass, equivalent stiffness, and equivalent resistance coefficient of the bar B. The true natural pulsatance ω_0 of the bar is then given by the equation

$$\omega_0^2 = s/m. \dots \dots \dots \quad (4.2)$$

If ξ be the displacement of the free end of the bar, $\dot{\xi}$ the velocity, then, when the pulsatance of the driving current is ω , the equation of motion of the bar is

$$[-m\omega^2 + j\omega r + (s - A)]\xi = Bi \dots \dots \quad (4.3)$$

in the notation of D. & J. (equation 14, p. 1031); A and B are factors connecting force with displacement and current respectively.

Since the velocity $\dot{\xi}$ is related to the displacement ξ by the equation $\dot{\xi} = j\omega\xi$, equation (4.3) may be written

$$[r + j(\omega m - s'/\omega)]\dot{\xi} = Bi \dots \dots \quad (4.4)$$

where $s' = (s - A)$ is the effective equivalent stiffness of the bar.

If the virtual value I of the current in the driving-coil be kept constant, then the peak value of the velocity of the bar is a maximum at the apparent resonant pulsatance ω_1 , given by

$$\omega_1^2 = s'/m = (s - A)m. \dots \dots \dots \quad (4.5)$$

In addition, as shown in D. & J. (equation 16, p. 1032), the pulsation at which the amplitude is a maximum can be identified with ω_1 because the resistance coefficient of the system is small.

Now, according to equation (9) of D. & J. (p. 1030), the "motional induced e.m.f." e_m in C_i due to the motion of the armature on the vibrating bar is given by

$$e_m = B_i \dot{\xi}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.6)$$

where B_i is the e.m.f. induced in C_i when the velocity of the armature is unity.

From equations (4.5) and (4.6), we obtain

$$e_m = \frac{BB_i i}{\left[r + j\left(\omega m - \frac{s'}{\omega}\right) \right]} \quad \dots \quad \dots \quad \dots \quad (4.7)$$

Let

$$f = \omega^2/\omega_1^2 = m\omega^2/s' \quad \text{and} \quad k = r/m\omega_1 \quad \dots \quad (4.8)$$

Then

$$r + j(\omega m - s'/\omega) = m\omega_1 [k + j(\sqrt{f} - 1/\sqrt{f})]. \quad \dots \quad (4.9)$$

and equation (4.7) becomes

$$e_m = \frac{BB_i}{m\omega_1} \cdot \frac{i}{\left[k + j\left(\sqrt{f} - \frac{1}{\sqrt{f}}\right) \right]} = \frac{BB_i}{m\omega_1} \cdot \frac{\left[k - j\left(\sqrt{f} - \frac{1}{\sqrt{f}}\right) \right] i}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}}\right)^2 \right]} \quad \dots \quad (4.10)$$

The virtual value E_m of the motional e.m.f. is clearly

$$E_m = \frac{BB_i I}{m\omega_1} \cdot \frac{1}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}}\right)^2 \right]} \quad \dots \quad (4.11)$$

It also follows that e_m consists of two components, e_{m1} in phase with i and e_{m2} lagging $\pi/2$ behind i , where

$$\left. \begin{aligned} e_{m1} &= \frac{BB_i}{m\omega_1} \cdot \frac{ki}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}}\right)^2 \right]}, \\ e_{m2} &= -\frac{BB_i}{m\omega_1} \cdot \frac{j\left(\sqrt{f} - \frac{1}{\sqrt{f}}\right)i}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}}\right)^2 \right]} \end{aligned} \right\} \quad \dots \quad (4.12)$$

At resonance ($\sqrt{f}=1$), the in-phase component e_{m1} is a maximum and the quadrature component e_{m2} vanishes.

For pulsatances below resonance, \sqrt{f} is less than 1 and $(\sqrt{f} - \frac{1}{\sqrt{f}})$ is negative, *i. e.*, the quadrature component leads i . For pulsatances above resonance, \sqrt{f} is greater than 1 and $(\sqrt{f} - \frac{1}{\sqrt{f}})$ is positive, *i. e.*, the quadrature component lags behind i .

When the bar is damped, *i. e.*, prevented from vibrating, the e.m.f. in the coil C_i is given by equation (4.1); when the bar is allowed to vibrate, the "free" e.m.f. e_f in the coil is the resultant of the "damped" e.m.f. of equation (4.1) and the "motional" e.m.f. of equation (4.10). e_f is thus the resultant of the following two components :

$$(i.) \quad \frac{BB_i}{m\omega_1} \cdot \frac{ki}{[k^2 + (\sqrt{f} - \frac{1}{\sqrt{f}})^2]} \quad \text{in phase with } i,$$

$$(ii.) - \left\{ \omega M + \frac{BB_i}{m\omega_1} \cdot \frac{(\sqrt{f} - \frac{1}{\sqrt{f}})}{[k^2 + (\sqrt{f} - \frac{1}{\sqrt{f}})^2]} \right\} i \quad \left\{ \begin{array}{l} \text{in quadra-} \\ \text{ture with } i \end{array} \right.$$

Hence, if E_f be the virtual value of the "free" e.m.f.,

$$E_f^2 = I^2 \left\{ \omega^2 M^2 + \frac{2BB_i M}{m} \cdot \frac{\sqrt{f}(\sqrt{f} - \frac{1}{\sqrt{f}})}{[k^2 + (\sqrt{f} - \frac{1}{\sqrt{f}})^2]} \right. \\ \left. + \frac{B^2 B_i^2}{m \omega_1^2} \cdot \frac{1}{[k^2 + (\sqrt{f} - \frac{1}{\sqrt{f}})^2]} \right\}, \quad (4.13)$$

whence, using equations (4.1) and (4.13)

$$E_f^2 = E_d^2 + \frac{2BB_i M I^2}{m} \cdot \frac{(f-1)}{[k^2 + (\sqrt{f} - \frac{1}{\sqrt{f}})^2]} + E_m^2. \quad (4.14)$$

The aim of the present work is the determination of Young's Modulus, and, as pointed out in D. & J. (p. 1084), for this purpose, there is little point in taking readings with large amplitudes of vibration of the bar, since the values of the resonant pulsatances obtained under these conditions are unimportant when extrapolating to the limiting conditions. Hence the discussion of equation (4.14) may be confined to the case where the amplitude of vibration is small. With this restriction E_d is large in comparison with E_m , and thus the E_m^2 term may be neglected in comparison with the first two terms. With this simplification, the equation becomes

$$E_f^2 - E_d^2 = \frac{2BB_iM}{m} \cdot \frac{(f-1)}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2 \right]} I^2. \quad (4.15)$$

It follows that at resonance ($f=1$), the "free" and "damped" e.m.f.'s are equal. This gives the following simple method of determining the apparent resonant pulsatance ω_1 :—Plot the "free" and "damped" e.m.f.'s against pulsatance; the pulsatance corresponding to the point of intersection of the two curves is the apparent resonant pulsatance, ω_1 .

From experimental results, it is possible to plot $(E_f^2 - E_d^2)$ against ω ; according to the above equation, the shape of this curve will be determined by the function F , where

$$F = \pm (f-1) / \left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2 \right]. \quad (4.16)$$

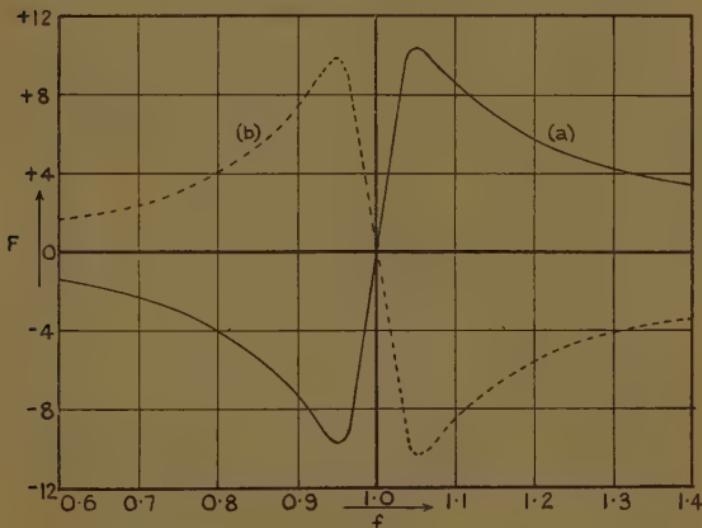
The \pm sign is necessary since the mutual inductance M between C and C_i can be positive or negative. The variation of F with f is shown in fig. 4, F being plotted as ordinate and f as abscissa. The continuous curve, (a), relates to F taken with the positive sign and the dotted curve, (b), to F taken with the negative sign. In plotting these curves, k has been assumed to be 0.05 (cf. D. & J., p. 1034).

In order to evaluate the "free" current in the coil C_i , it is necessary to know the motional impedance Z_m of C_i . Z_m is a function of ω and its value is given by the equation

$$Z_m = B^2 / \left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2 \right] m\omega_1$$

(v. D. & J., equation 20, p. 1035). Z_m is a maximum when $f=1$ and, for small amplitudes, the values obtained experimentally for Z_m at resonance scarcely ever exceed 0.5 ohm (cf. D. & J., figs. 9 b and 10 b, pp. 1073 and 1074). In the experiments described below, the total impedance of the coil C_i , together with that of the a.c. milliammeter used for measuring the current in the circuit, amounted to about 700 ohms, and hence the maximum error committed in neglecting Z_m in comparison with Z_i (the damped impedance of the coil) will not exceed 1 in 1400. The impedance of C_i when the reed is vibrating may

Fig. 4.



F—f Curves ; (a), F taken with positive sign,
(b), F taken with negative sign.

therefore be taken to be equal to its damped impedance Z_i . The "free" current I_f in the coil is thus equal to E_f/Z_i and, from equations (4.1) and 4.15), we obtain

$$I_f^2 - I_d^2 = \frac{E_f^2 - E_d^2}{Z_i^2} = \frac{2BB_i MI^2}{mZ_i^2} \cdot \frac{(f-1)}{\left[k^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2 \right]}, \quad \dots \quad (4.17)$$

showing that if $(I_f^2 - I_d^2)$ is plotted against f , the curve obtained will be similar to that connecting $(E_f^2 - E_d^2)$

and f . In addition, ω_1 can be determined by finding the pulsatance at which I_f and I_d are equal.

It is possible to evaluate k from the curves connecting $(E_f^2 - E_d^2)$ or $(I_f^2 - I_d^2)$ with f . On differentiating equation (4.16) with respect to f , it is found that the turning points of these curves occur at $f = 1/(1 \pm k)$; the values of k can therefore be deduced from the values of f at the maxima and minima of the curves.

(d) *Extension of the Theory to Include Eddy-Current and Hysteresis Effects.*

The driving-coil, C, of fig. 3 is placed on a solid soft-iron core, and thus the simple theory given above must be extended so as to include the effects of hysteresis and eddy-currents in this core. The necessary modifications, because of the presence of eddy-current and hysteresis loss in the core, has been shown to lead to the following equation for the motion of the bar :

$$(-m\omega^2 + j\omega r_e + s_e)\xi = (a_1 - ja_2)i, \quad \dots \quad (4.18)$$

where

r_e = the effective equivalent resistance coefficient of the bar ;

s_e = the effective equivalent stiffness coefficient of the bar ; and

a_1, a_2 = the in-phase and quadrature components of the force factor of the driving coil ;

(v. D. & J., equation 42, p. 1047).

Comparing this equation with the corresponding equation (4.3) which applies in the absence of eddy-currents and hysteresis, it is clear that the main difference lies in the fact that, in the present case, the vibromotive force lags behind the driving-current, the angle of lag being $\tan^{-1}(a_2/a_1)$.

In terms of the velocity $\dot{\xi}$, equation (4.18) may be written

$$\left[r_e + j \left(\omega m - \frac{s_e}{\omega} \right) \right] \dot{\xi} = (a_1 - ja_2)i. \quad \dots \quad (4.19)$$

If the virtual value, I , of the driving current be maintained constant, then the peak value of the velocity is a maximum at the pulsatance ω_r , where $\omega_r^2 = s_e/m$;

as before, ω_r may also be identified with the pulsatance at which the amplitude is a maximum.

The core of the coil C_i is non-conducting and non-magnetic, and thus the e.m.f., e_m , induced in C_i by the motion of the bar will be given by $B_i \dot{\xi}$ as before [v. equation (4.6)].

Writing

$$f = \frac{\omega^2}{\omega_r^2} = \frac{m\omega^2}{s_e}; \quad k_e = r_e/m\omega. \quad \dots \quad (4.20)$$

and eliminating $\dot{\xi}$ from equations (4.6) and (4.19), we obtain

$$e_m = -\frac{B_i(a_1 - ja_2)i}{r_e + j\left(\omega m - \frac{s_e}{\omega}\right)},$$

which reduces to

$$e_m = \frac{B_i i}{m\omega_r} \times \left\{ \frac{\left[a_1 k_e - a_2 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right) \right] - j \left[a_2 k_e - a_1 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right) \right]}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\}. \quad \dots \quad (4.21)$$

The in-phase component of e_m is thus

$$\frac{B_i i}{m\omega_r} \left\{ \frac{a_1 k_e - a_2 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\} \quad \dots \quad (4.22)$$

and the quadrature component

$$-\frac{B_i i}{m\omega_r} \left\{ \frac{a_2 k_e + a_1 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\}. \quad \dots \quad (4.23)$$

In calculating the "free" and "damped" e.m.f.'s (e_f and e_d) in the coil C_i , it must be remembered that eddy-currents and hysteresis in the core will affect the "damped" e.m.f., since the flux due to the current in C

will no longer be in phase with the current. This is equivalent to regarding the mutual inductance M as a complex quantity of the form $(M_1 + jM_2)$; hence

$$e_d = -j\omega Mi = \omega i(M_2 - jM_1),$$

or, in terms of virtual values,

$$E_d^2 = \omega^2(M_1^2 + M_2^2)I^2. \dots \quad (4.24)$$

It follows that the in-phase component of e_f is $\omega M_2 i$ plus the in-phase component of e_m given by equation (4.22); the quadrature component of e_f is similarly $-\omega M_1 i$ plus the quadrature component of e_m given by equation (4.23). The virtual value, E_f , of the "free" e.m.f. thus is given by

$$\frac{E_f^2}{I^2} = \left[\omega M_2 + \frac{B_i}{m\omega_r} \left\{ \frac{a_1 k_e - a_2 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\} \right]^2 + \left[\omega M_1 + \frac{B_i}{m\omega_r} \left\{ \frac{a_2 k_e + a_1 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\} \right]^2. \quad (4.25)$$

If the motional e.m.f. E_m be neglected in comparison with E_d as before, then this equation reduces to

$$\frac{E_f^2 - E_d^2}{I^2} = \frac{2B_i\sqrt{f}}{m} \left\{ \frac{M_1 \left[a_2 k_e + a_1 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right) \right] + M_2 \left[a_1 k_e - a_2 \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right) \right]}{k_e^2 + \left(\sqrt{f} - \frac{1}{\sqrt{f}} \right)^2} \right\}. \quad (4.26)$$

The condition that the "free" and "damped" e.m.f.'s should be equal is thus

$$\left. \begin{aligned} \sqrt{f} - \frac{1}{\sqrt{f}} &= \gamma, \\ \gamma &= k_e \frac{(M_1 a_2 + M_2 a_1)}{(M_2 a_2 - M_1 a_1)}. \end{aligned} \right\} \quad \dots \quad (4.27)$$

where

Equation (4.27) may be regarded as a quadratic equation in \sqrt{f} , whose solution is

$$\sqrt{f} = \frac{\gamma}{2} \pm \left(1 + \frac{\gamma^2}{4}\right)^{\frac{1}{2}}.$$

The appropriate sign before the root is clearly the positive sign ; expanding this root, we therefore obtain

$$\sqrt{f} = \frac{\omega}{\omega_r} = 1 + \frac{\gamma}{2} + \frac{\gamma^2}{8} + \dots \dots \quad (4.28)$$

It is thus clear that the effect of eddy-currents and hysteresis is to cause $(E_f^2 - E_d^2)$ to vanish at a pulsatance other than ω_r ; the difference between ω_r and the pulsatance at which $(E_f^2 - E_d^2)$ vanishes is determined by the magnitude of γ . It is possible to determine the values of k_e and (a_2/a_1) from resonance and motional impedance curves, but it seems difficult to estimate the relative values of M_1 and M_2 , and therefore to determine the order of magnitude of γ . Similar considerations refer to the pulsatance at which $(I_f^2 - I_d^2)$ vanishes.

It was therefore decided to investigate experimentally the difference between ω_r and the pulsatance at which $(E_f^2 - E_d^2)$ vanishes (or at which I_f and I_d are equal), and the results are given in the next section ; it appears from a large number of experiments that this difference does not exceed 1 in 500 under the conditions prevailing in these experiments.

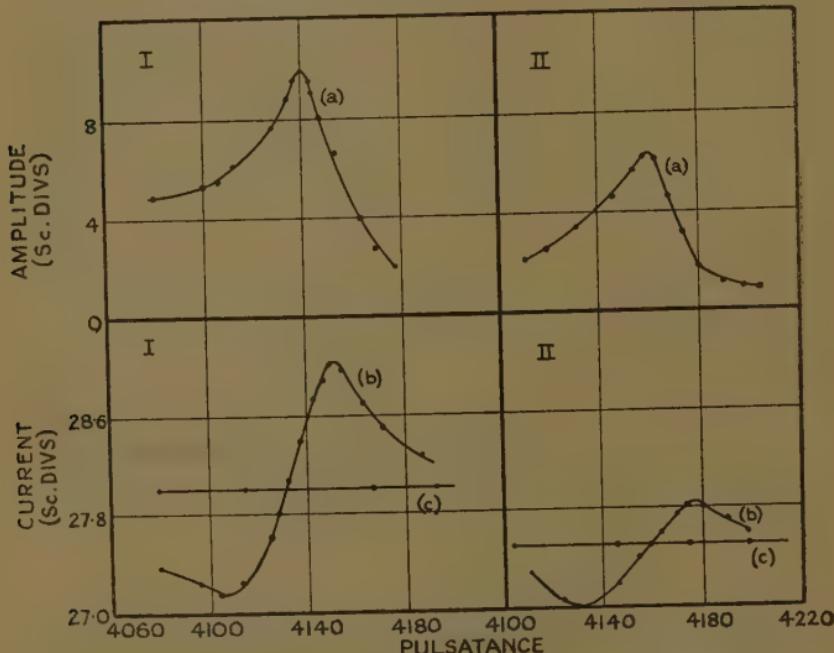
(e) *Experimental Details and Results.*

To test the applicability of the above method for the determination of the resonant pulsatance ω_r , the arrangement shown in D. & J., fig. 1, p. 1055, was used with the addition of the coil C of the preceding sections. This coil, wound on an ebonite former, was supported on an ebonite rod so as to lie immediately to the right of the vibrating bar ; the coil was placed so that its centre was on the axis of the remainder of the apparatus. The inside diameter of the channel on which the coil was wound was 2 cm., the outside diameter being 3.5 cm., and the width of the channel 1.0 cm. ; the coil consisted of about 1900 turns of no. 26 S.W.G. enamelled copper wire. The resistance of the coil was 27.4 ohm and its inductance 80 mH.

The e.m.f. and current in this coil were measured by means of a Cambridge Instrument Co. versatile galvanometer fitted an A.C. range-box; in the following experiments, the instrument was used on the 0-1 mA. range. The resistance of the instrument on this range was found to be 548.7 ohms and its inductance 1.25 mH.

A large number of sets of readings of amplitudes, I_f , I_d , and pulsatances relating to different bars, different loads, and different lengths were taken with this apparatus;

Fig. 5.



Typical experimental curves showing the relation between :

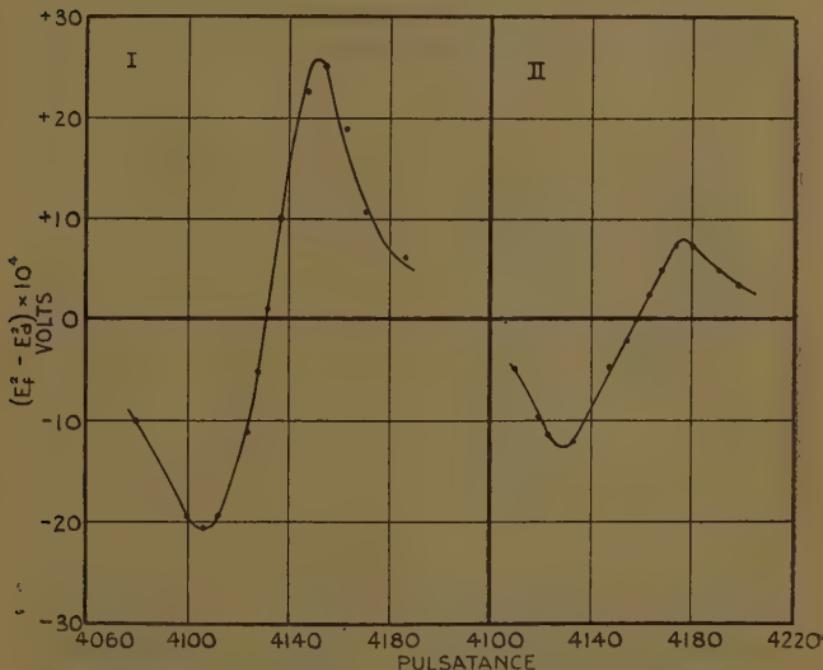
- (a) amplitude ; (b) "free" current ;
- (c) "damped" current ; and pulsatance.

two typical sets of results are shown in figs. 5 and 6. The curves marked "I" in each figure relate to a comparatively large amplitude of vibration, whilst the curves marked "II" related to a smaller amplitude. In both figures, pulsatance ω is plotted as abscissa. In fig. 5, the upper curves "a" show the relation between amplitude and pulsatance, the amplitude being measured in microscope scale-divisions (1 scale-division = 0.107 mm.); the two lower curves, "b" and "c," show the variation

of I_f and I_d respectively with pulsatance; the currents are given in galvanometer scale-divisions (1 scale division = 0.01 mA.).

The curves in fig. 6 show the relation between $(E_f^2 - E_d^2)$ and pulsatance; the values of E_f and E_d have been calculated from the values of I_f and I_d and the impedance of the coil-circuit (including the galvanometer). E_f and E_d are given in volts and the motional impedance of the coil has been neglected for the reason given in section (b).

Fig. 6.



Experimental curves (derived from fig. 5) showing the relation between $(E_f^2 - E_d^2)$ and pulsatance.

In the curves relating to set I, the resonance pulsatance given by the amplitude curve is 4140, whilst that deduced from the $(E_f^2 - E_d^2)$ curve is 4132; for set II, the corresponding resonant pulsatances are 4160 and 4156 respectively. For these two sets of results, the maximum discrepancy between the resonant pulsatances found by the two methods is about 1 in 500, and in no case was the discrepancy greater than this; it should also be noted that the discrepancy is greater with the greater

amplitude, and this was found to be generally true. It is shown later that, with the bars used in the present work, an accuracy of 1 in 1000 in the measurement of resonant pulsatance is necessary ; this method cannot, therefore, be used since it does not give this degree of accuracy.

Finally, it may be noted that the shape of the $(E_f^2 - E_d^2)$ curves found experimentally (fig. 6) is very similar to that of the theoretical curve (fig. 4).

(f) *Comparison of the Various Methods for the Determination of Resonant Pulsatances.*

It remains to compare the relative merits of the three methods described for the measurement of resonant pulsatances.

As regards convenience, there is little to choose between the amplitude and the "free" e.m.f. methods, whilst the motional impedance method is much more inconvenient, both as regards the experimental work and subsequent calculations.

As regards accuracy, under favourable conditions, resonant pulsatances can be determined to within 1 in 1000 by the amplitude method and the motional impedance method, whilst the accuracy of the "free" e.m.f. method has been shown above to exceed 1 in 500. On account of other factors (v. Section VII.) an accuracy of 1 in 1000 is necessary, and thus adequate accuracy is given only by the first two methods.

The third point to be considered is the relative sensitivity of the three methods ; this is important, since, with the extrapolation method used, the resonant pulsatances for small amplitudes of vibration are more important than those for large amplitudes. With the motional impedance method, the measurement of motional impedance tends to become difficult and inaccurate when the amplitude becomes small. The same is true, but to a smaller extent, for the "free" e.m.f. method. On the other hand, the amplitude method can be used when the amplitude is small, even when it is too small to be measured accurately ; with such small amplitudes, the motional impedance and "free" e.m.f. methods become inapplicable.

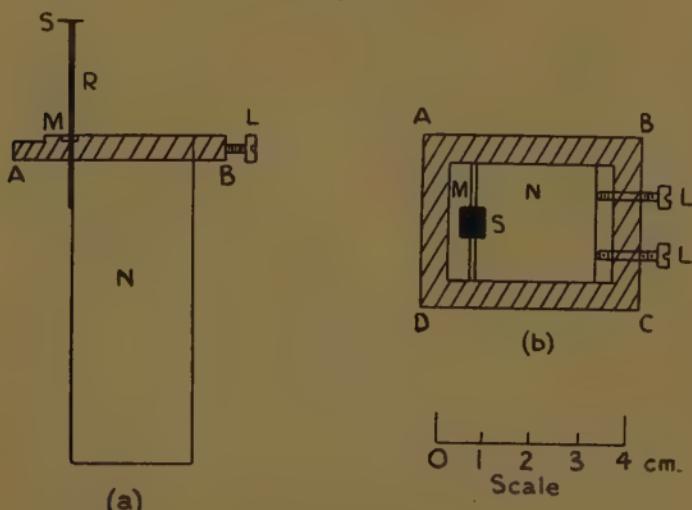
In practice, it is found that the amplitude method is the one which gives greatest accuracy in the extrapolated

resonant frequencies; this method is also the most convenient method and thus, in the experimental results given below, resonant frequencies have been determined by the amplitude method.

V. DESCRIPTION OF THE APPARATUS AND THE METHOD.

The experimental arrangement and procedure are identical with those used by D. & J., with the exception of the arrangement used for clamping the lower end of the vibrating bar. The arrangement used in the present work for this purpose is shown in section in fig. 7 a and in plan in fig. 7 b.

Fig. 7.



The arrangement used for clamping the end of the bar.

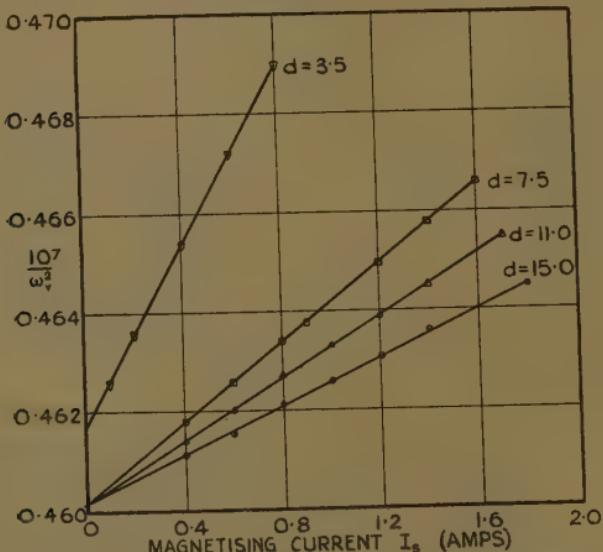
The bar R is clamped between a large steel block N and a steel strip M, the bar and M being firmly held in position against the block by means of the frame ABCD. The block N consists of a piece of steel 7 cm. long and 2.5 cm. square section; the strip M is 2.5 cm. long, 0.5 cm. wide, and 0.5 cm. thick. The thickness of the frame ABCD is 0.5 cm., the side AB of the frame being 4.5 cm. and the side AD 3.6 cm. The frame is fitted with two steel screws LL as shown.

The upper faces of M and N are ground so that they can be adjusted to lie in the same plane; the adjacent faces of M and N are also ground, the angle between these surfaces and the upper faces being accurately 90° .

The method of assembly is as follows:—The bar R is placed between M and N; the frame ABCD is then placed in position. The vibrating bar R is then adjusted so that its length is perpendicular to the upper surface of N and, finally, M is adjusted so that its upper surface is in the same plane as the upper surface of N. The screws LL are then tightened. This arrangement has been found to be convenient and reliable.

The vibrating length of the bar is the distance between the plane of the upper surfaces of M and N and the centre of the stalloy armature S. This distance is

Fig. 8.



Extrapolating curves annealed Ferry Magnetizing Current I_s .

measured with a travelling microscope, and in order to be able to focus on the upper surfaces of M and N, a portion of the frame ABCD is cut away, as shown in the diagram.

The quantities which must be determined experimentally in order to evaluate Young's Modulus E have been given in the introductory section, and it remains only to describe the method used to extrapolate the apparent resonant pulsatances ω_r to the true resonant pulsatance ω_0 . The theoretical basis of the method of extrapolation has been given in Sections II. and III., and the method itself can be best explained by reference to fig. 8 which shows

a set of curves obtained for a typical imperfectly elastic alloy—annealed Ferry. In this diagram, the magnetizing current I_s (in amps.) is plotted as abscissa and $(1/\omega_r^2)$, where ω_r is the apparent resonant pulsatance, as ordinate. All four curves were obtained with a bar of length 2.463 cm. for which c was 0.04569; this bar was used for preliminary experiments and is not the one used for the results in Section VI. Preliminary experiments had shown that the magnitude of the driving current had a negligible effect on the value of the apparent resonant pulsatance with the small values of amplitude of vibration which were used; for this reason, the driving current was kept constant at about 6 mA. for all four curves.

The upper curve shows the relation between $(1/\omega_r^2)$ and I_s when the air-gap, d , between the pole-piece and the adjacent edge of the stalloy armature was constant and equal to 3.5 microscope scale-divisions (1 scale-division = 1.07 mm.). It is seen that, as the value of I_s decreases, the value of $1/\omega^2$ decreases also; for small values of the current, the relation between $1/\omega_r^2$ and I_s is approximately linear.

The three lower curves were obtained with air-gaps of 7.5, 11.0, and 15.0 scale-divisions respectively. These curves are seen to be straight lines for all values of the direct current used in the experiments and it is thus easy to extrapolate accurately to zero direct current. This may be done graphically or by using some method of computation, such as the method of least squares or Awbery's method (7). It is seen from the diagram that the three lower curves extrapolate for zero direct current to practically the same value of $1/\omega_r^2$, namely, 0.4602×10^{-7} ; the value of ω_0 is therefore $(0.4602 \times 10^{-7})^{-\frac{1}{2}} = 4661$. It is also clear from the diagram that graphical extrapolation is sufficiently accurate—in fact, the discrepancy between the extrapolated pulsatance derived from each of the three lower curves in fig. 8 is less than 1 in 2500.

The experimental results are given in the next section; in the case of each alloy, a set of extrapolating curves, similar to the set given above, was obtained for each of three different lengths of a single bar, except in the case of annealed nickel and annealed monel, in which cases two different bars were used. In each instance,

the extrapolated true resonant pulsatances obtained with different air-gaps were in agreement to within 1 in 2000. In order to save space, these extrapolating curves are not given, since they are all of the same general form as those given in fig. 8. It should, however, be noted that, in general, the slope of the curves for ferromagnetic alloys is much less than the slopes of the curves for non-magnetic alloys; in some instances, *e.g.*, annealed nickel, $1/\omega_r^2$ was independent of the value of the steady polarizing current.

The results given below were obtained at room temperature (approximately $19^\circ\text{C}.$); it was unnecessary to keep the temperature accurately constant, since the temperature variation of Young's Modulus is very small, *e.g.*, in the case of monel, an increase of temperature $1^\circ\text{C}.$ is accompanied by a decrease of only 1 in 1200 in Young's Modulus.

Since the method is a dynamical method, the moduli given are adiabatic moduli.

VI. EXPERIMENTAL RESULTS.

The notation used in this Section is identical with that of Section I. where the theory of the method is given. In addition, let b denote the width of the bar, d its thickness, and let a quantity Z be defined by the equation

$$Z = \frac{l^2 n_0}{z^2} \dots \dots \dots \quad (6.1)$$

From equation (1.1), it follows that

$$E = \frac{4\pi^2 \rho Z^2}{k^2} = \frac{48\pi^2 \rho Z^2}{d^2}, \dots \dots \quad (6.2)$$

whence it follows that, for a bar of a given material and of a given thickness, Z should be constant. In the following results, the value of Z is calculated for each of the three lengths used for each bar; finally, E is calculated from equation (6.2), using the mean value of Z .

(1) NICKEL.

(a) Annealed Specimens.

Specimen No. 1.

$M=0.04110 \text{ gm.}; b=0.646 \text{ cm.};$

$d=0.0765 \text{ cm.}; \rho=8.910 \text{ gm./c.c.}$

1st length : $l=2.568$ cm. ;

$$c=0.03663 ; \frac{1}{\omega_0^2}=0.3819 \times 10^{-7} ; Z=1636.$$

2nd length : $l=2.598$ cm. ;

$$c=0.03622 ; \frac{1}{\omega_0^2}=0.3966 \times 10^{-7} ; Z=1638.$$

3rd length : $l=2.775$ cm. ;

$$c=0.3453 ; \frac{1}{\omega_0^2}=0.5322 \times 10^{-7} ; Z=1638.$$

The mean value of Z is 1637.3, giving

$$\underline{E=19.37 \times 10^{11} \text{ dynes/sq. cm.}}$$

Specimen No. 2.

$$M=0.0623 \text{ gm.} ; b=0.646 \text{ cm.} ; d=0.0760 \text{ cm.}$$

1st length : $l=2.510$ cm. ;

$$c=0.03021 ; \frac{1}{\omega_0^2}=0.3768 \times 10^{-7} ; Z=1628.$$

2nd length : $l=2.592$ cm. ;

$$c=0.02978 ; \frac{1}{\omega_0^2}=0.3921 \times 10^{-7} ; Z=1629.$$

3rd length : $l=2.644$ cm. ;

$$c=0.02751 ; \frac{1}{\omega_0^2}=0.4441 \times 10^{-7} ; Z=1629.$$

The mean value of Z is 1628.7, giving

$$\underline{E=19.37 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.1105 \text{ gm.} ; b=0.6430 \text{ cm.} ;$$

$$d=0.727 \text{ cm.} ; \rho=8.831 \text{ gm./c.c.}$$

1st length : $l=2.321$ cm. ;

$$c=0.1153 ; \frac{1}{\omega_0^2}=3.689 \times 10^{-8} ; Z=1549.$$

2nd length : $l=2.597$ cm. ;

$$c=0.1031 ; \frac{1}{\omega_0^2}=5.339 \times 10^{-8} ; Z=1551.$$

3rd length : $l=2.790$ cm. ;

$$c=0.9595 ; \frac{1}{\omega_0^2}=7.157 \times 10^{-8} ; Z=1549.$$

The mean value of Z is 1549.7, giving

$$\underline{E=19.18 \times 10^{11} \text{ dynes/sq. cm.}}$$

(2) MONEL.

(a) *Annealed Specimens.*

Specimen No. 1.

$$M=0.0710 \text{ gm.}; b=0.6465 \text{ cm.};$$

$$d=0.0769 \text{ cm.}; \rho=8.814 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.280 \text{ cm.};$$

$$c=0.07106; \frac{1}{\omega_0^2}=0.3385 \times 10^{-7}; Z=1454.$$

$$2\text{nd length: } l=2.410 \text{ cm.};$$

$$c=0.06725; \frac{1}{\omega_0^2}=0.4209 \times 10^{-7}; Z=1454.$$

$$3\text{rd length: } l=2.515 \text{ cm.};$$

$$c=0.06442; \frac{1}{\omega_0^2}=0.4896 \times 10^{-7}; Z=1453.$$

The mean value of Z is 1453.7, giving

$$\underline{E=14.92 \times 10^{11} \text{ dynes/sq. cm.}}$$

Specimen No. 2.

$$M=0.0511 \text{ gm.}; b=0.646 \text{ cm.};$$

$$d=0.081 \text{ cm.}; \rho=8.814 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.390 \text{ cm.};$$

$$c=0.04641; \frac{1}{\omega_0^2}=2.901 \times 10^{-8}; Z=1535.$$

$$2\text{nd length: } l=2.428 \text{ cm.};$$

$$c=0.04381; \frac{1}{\omega_0^2}=3.621 \times 10^{-8}; Z=1534.$$

$$3\text{rd length: } l=2.747 \text{ cm.};$$

$$c=0.04038; \frac{1}{\omega_0^2}=4.977 \times 10^{-8}; Z=1534.$$

The mean value of Z is 1534.3, giving

$$\underline{E=14.95 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.0873 \text{ gm.}; b=0.643 \text{ cm.};$$

$$d=0.0548 \text{ cm.}; \rho=8.892 \text{ gm./c.c.}$$

1st length : $l=2.013$ cm. ;

$$c=0.1385 ; \frac{1}{\omega_0^2}=4.636 \times 10^{-8} ; Z=1054.$$

2nd length : $l=2.164$ cm. ;

$$c=0.1287 ; \frac{1}{\omega_0^2}=5.984 \times 10^{-8} ; Z=1053.$$

3rd length : $l=2.232$ cm. ;

$$c=0.1248 ; \frac{1}{\omega_0^2}=6.609 \times 10^{-8} ; Z=1054.$$

The mean value of Z is 1053.7, giving

$$\underline{E=15.57 \times 10^{11} \text{ dynes/sq. cm.}}$$

(3) 48 per cent. NICKEL IRON.

(a) *Annealed Specimen.*

$$M=0.0889 \text{ gm.} ; b=0.6418 \text{ cm.} ;$$

$$d=0.0783 \text{ cm.} ; \rho=8.256 \text{ gm./c.c.}$$

1st length : $l=2.260$ cm. ;

$$c=0.09481 ; \frac{1}{\omega_0^2}=3.747 \times 10^{-8} ; Z=1464.$$

2nd length : $l=2.700$ cm. ;

$$c=0.07936 ; \frac{1}{\omega_0^2}=6.748 \times 10^{-8} ; Z=1459.$$

3rd length : $l=2.765$ cm. ;

$$c=0.07750 ; \frac{1}{\omega_0^2}=7.313 \times 10^{-8} ; Z=1461.$$

The mean value of Z is 1461.3, giving

$$\underline{E=13.65 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.0892 \text{ gm.} ; b=0.6395 \text{ cm.} ;$$

$$d=0.0772 \text{ cm.} ; \rho=8.179 \text{ gm./c.c.}$$

1st length : $l=2.584$ cm. ;

$$c=0.08549 ; \frac{1}{\omega_0^2}=5.214 \times 10^{-8} ; Z=1534.$$

2nd length : $l=2.656$ cm. ;

$$c=0.08317 ; \frac{1}{\omega_0^2}=5.788 \times 10^{-8} ; Z=1533.$$

3rd length : $l=2.868$ cm. ;

$$c=0.07702; \frac{1}{\omega_0^2}=7.721 \times 10^{-8}; Z=1534.$$

The mean value of Z is 1533.7, giving

$$\underline{E=15.29 \times 10^{11} \text{ dynes/sq. cm.}}$$

(4) FERRY.

(a) *Annealed Specimen.*

$$M=0.1720 \text{ gm.}; b=0.6410 \text{ cm.};$$

$$d=0.0781 \text{ cm.}; \rho=8.886 \text{ gm./c.c.}$$

$$1\text{st length : } l=2.774 \text{ cm.};$$

$$c=0.1394; \frac{1}{\omega_0^2}=9.076 \times 10^{-8}; Z=1448.$$

$$2\text{nd length : } l=2.635 \text{ cm.};$$

$$c=0.1468; \frac{1}{\omega_0^2}=7.496 \times 10^{-8}; Z=1450.$$

$$3\text{rd length : } l=2.450 \text{ cm.};$$

$$c=0.1578; \frac{1}{\omega_0^2}=5.572 \times 10^{-8}; Z=1450.$$

The mean value of Z is 1449.3, giving

$$\underline{E=14.54 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.1255 \text{ gm.}; b=0.644 \text{ cm.};$$

$$d=0.0785 \text{ cm.}; \rho=8.869 \text{ gm./c.c.}$$

$$1\text{st length : } l=2.347 \text{ cm.};$$

$$c=0.1193; \frac{1}{\omega_0^2}=4.478 \times 10^{-8}; Z=1433.$$

$$2\text{nd length : } l=2.859 \text{ cm.};$$

$$c=0.0979; \frac{1}{\omega_0^2}=8.969 \times 10^{-8}; Z=1434.$$

$$3\text{rd length : } l=2.548 \text{ cm.};$$

$$c=0.1099; \frac{1}{\omega_0^2}=6.106 \times 10^{-8}; Z=1431,$$

giving, as the mean value of Z 1432.7, which gives

$$\underline{E=14.01 \times 10^{11} \text{ dynes/sq. cm.}}$$

(5) BRIGHTRAY.

(a) Annealed Specimen.

$$M=0.0637 \text{ gm.}; b=0.6366 \text{ cm.};$$

$$d=0.0784 \text{ cm.}; \rho=8.318 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.911 \text{ cm.};$$

$$c=0.05273; \frac{1}{\omega_0^2}=5.552 \times 10^{-8}; Z=1772.$$

$$2\text{nd length: } l=2.678 \text{ cm.};$$

$$c=0.05730; \frac{1}{\omega_0^2}=4.028 \times 10^{-8}; Z=1774.$$

$$3\text{rd length: } l=2.482 \text{ cm.};$$

$$c=0.06182; \frac{1}{\omega_0^2}=3.111 \times 10^{-8}; Z=1774.$$

The mean value of Z is 1773.3, giving

$$\underline{E=20.17 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) Unannealed Specimen.

$$M=0.0712 \text{ gm.}; b=0.6393 \text{ cm.};$$

$$d=0.0787 \text{ cm.}; \rho=8.364 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.646 \text{ cm.};$$

$$c=0.06396; \frac{1}{\omega_0^2}=4.321 \times 10^{-8}; Z=1703.$$

$$2\text{nd length: } l=2.488 \text{ cm.};$$

$$c=0.06800; \frac{1}{\omega_0^2}=3.443 \times 10^{-8}; Z=1704.$$

$$3\text{rd length: } l=2.367 \text{ cm.};$$

$$c=0.07148; \frac{1}{\omega_0^2}=2.840 \times 10^{-8}; Z=1704.$$

The mean value of Z is 1703.7, giving

$$\underline{E=18.61 \times 10^{11} \text{ dynes/sq. cm.}}$$

(6) GLOWRAY.

(a) Annealed Specimen.

$$M=0.1375 \text{ gm.}; b=0.6450 \text{ cm.};$$

$$d=0.0800 \text{ cm.}; \rho=8.193 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.625 \text{ cm.};$$

$$c=0.1239; \frac{1}{\omega_0^2}=5.131 \times 10^{-8}; Z=1687.$$

2nd length : $l=2.768$ cm. ;

$$c=0.1175 ; \frac{1}{\omega_0^2}=6.237 \times 10^{-8} ; Z=1686.$$

3rd length : $l=2.782$ cm. ;

$$c=0.1169 ; \frac{1}{\omega_0^2}=6.343 \times 10^{-8} ; Z=1686.$$

The mean value of Z is 1686.3, giving

$$\underline{E=17.23 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.1310 \text{ gm.} ; b=0.6410 \text{ cm.} ;$$

$$d=0.0735 \text{ cm.} ; \rho=8.241 \text{ gm./c.c.}$$

1st length : $l=2.474$ cm. ;

$$c=0.1364 ; \frac{1}{\omega_0^2}=3.701 \times 10^{-8} ; Z=1794.$$

2nd length : $l=2.640$ cm. ;

$$c=0.1278 ; \frac{1}{\omega_0^2}=4.665 \times 10^{-8} ; Z=1796.$$

3rd length : $l=2.897$ cm. ;

$$c=0.1165 ; \frac{1}{\omega_0^2}=6.424 \times 10^{-8} ; Z=1794.$$

The mean value of Z is 1794.7, giving

$$\underline{E=23.33 \times 10^{11} \text{ dynes/sq. cm.}}$$

(7) DULLRAY.

(a) *Annealed Specimen.*

$$M=0.0650 \text{ gm.} ; b=0.6400 \text{ cm.} ;$$

$$d=0.0810 \text{ cm.} ; \rho=8.172 \text{ gm./c.c.}$$

1st length : $l=2.893$ cm. ;

$$c=0.04936 ; \frac{1}{\omega_0^2}=4.936 \times 10^{-8} ; Z=1614.$$

2nd length : $l=2.692$ cm. ;

$$c=0.05562 ; \frac{1}{\omega_0^2}=4.027 \times 10^{-8} ; Z=1617.$$

3rd length : $l=2.487$ cm. ;

$$c=0.06140 ; \frac{1}{\omega_0^2}=3.208 \times 10^{-8} ; Z=1615.$$

The mean value of Z is 1615.3, giving

$$\underline{E=15.38 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.0684 \text{ gm.}; b=0.6397 \text{ cm.};$$

$$d=0.0810 \text{ cm.}; \rho=8.098 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.507 \text{ cm.};$$

$$c=0.06502; \frac{1}{\omega_0^2}=4.140 \times 10^{-8}; Z=1570.$$

$$2\text{nd length: } l=2.398 \text{ cm.};$$

$$c=0.06798; \frac{1}{\omega_0^2}=3.548 \times 10^{-8}; Z=1568.$$

$$3\text{rd length: } l=2.227 \text{ cm.};$$

$$c=0.07020; \frac{1}{\omega_0^2}=2.719 \times 10^{-8}; Z=1568.$$

The mean value of Z is 1568.7, giving

$$\underline{E=14.42 \times 10^{11} \text{ dynes/sq. cm.}}$$

(8) 20 per cent. NICKEL SILVER.

(a) *Annealed Specimen.*

$$M=0.0645 \text{ gm.}; b=0.643 \text{ cm.};$$

$$d=0.0790 \text{ cm.}; \rho=8.686 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.307 \text{ cm.};$$

$$c=0.06339; \frac{1}{\omega_0^2}=4.103 \times 10^{-8}; Z=1374.$$

$$2\text{nd length: } l=2.484 \text{ cm.};$$

$$c=0.05886; \frac{1}{\omega_0^2}=5.060 \times 10^{-8}; Z=1374.$$

$$3\text{rd length: } l=2.664 \text{ cm.};$$

$$c=0.05488; \frac{1}{\omega_0^2}=6.738 \times 10^{-8}; Z=1373.$$

The mean value of Z is thus 1373.7, giving

$$\underline{E=12.45 \times 10^{11} \text{ dynes/sq. cm.}}$$

(b) *Unannealed Specimen.*

$$M=0.0744 \text{ gm.}; b=0.6400 \text{ cm.};$$

$$d=0.0780 \text{ cm.}; \rho=8.764 \text{ gm./c.c.}$$

$$1\text{st length: } l=2.644 \text{ cm.};$$

$$c=0.06432; \frac{1}{\omega_0^2}=6.992 \times 10^{-8}; Z=1320.$$

2nd length : $l=2.273$ cm. ;

$$c=0.07482; \frac{1}{\omega_0^2}=4.207 \times 10^{-8}; Z=1321.$$

3rd length : $l=2.342$ cm. ;

$$c=0.07261; \frac{1}{\omega_0^2}=4.308 \times 10^{-8}; Z=1323.$$

The mean value of Z is thus 1321.3, giving

$$\underline{E=11.98 \times 10^{11} \text{ dynes/sq. cm.}}$$

The above results, together with the previous results for the resistivities of the alloys⁽¹⁾ are summarized in the following table :—

Substance.	Density (gm./c.c.).	Young's Modulus (dynes/sq. cm.).	Resistivity (μ ohm. cm.).
Nickel (A.)	8.910	19.37×10^{11}	10.84
(U.)	8.831	19.18	10.57
Monel (A.)	8.814	14.94	46.8
(U.)	8.892	15.57	48.7
48 per cent. nickel iron { (A.)	8.256	13.65	50.6
(U.)	8.179	15.29	52.5
Ferry (A.)	8.886	14.54	49.9
(U.)	8.869	14.01	50.8
Brightray (A)	8.318	20.17	109.6
(U.)	8.364	18.61	99.0
Glowray (A.)	8.193	17.23	109.9
(U.)	8.241	23.33	103.5
Dullray (A.)	8.172	15.38	93.0
(U.)	8.098	14.42	91.6
20 per cent. nickel-silver { (A.)	8.686	12.45	30.9
(U.)	8.764	11.88	29.5

Note.—A. Annealed Specimen.

U. Unannealed Specimen.

VII. DISCUSSION OF THE RESULTS AND COMPARISON WITH THE RESULTS OF PREVIOUS INVESTIGATIONS.

In considering the accuracy of the results, it is seen from equation (1.1) that the expression for E involves :

- (i.) the fourth power of the length of the bar ;
- (ii.) the square of the natural pulsatance ω_0 ;
- (iii.) the density of the bar ;
- (iv.) the reciprocal of the square of the thickness of the bar ;
- and (v.) the reciprocal of the fourth power of the vibration parameter z .

The lengths used in these experiments are of the order of 2.5 cm., and with care, it is possible to determine the length to within 0.001 cm.; the probable error in E due to errors in measuring the length is thus of the order of 1 in 600.

The natural pulsatance ω_0 can be determined to an accuracy of 1 in 1000, which implies that the error in E due to the error in determining ω_0 is of the order of 1 in 500.

The density of the materials can be determined to within 1 in 1000; this corresponds to an error of 1 in 1000 in E .

The thickness of the bars are of the order of 0.08 cm.; this quantity can be measured to within 0.0002 cm. An error of 0.0002 cm. in the measurement of the thickness leads to an error of 1 in 200 in E .

Turning next to the effect of the errors in the vibration parameter z , inspection of the values of z for different lengths of a given bar shows that under the experimental conditions, the variation in the value of z is small. For example, in the case of the annealed specimen of Brightray, the value of z corresponding to a length of 2.678 cm. is 1.780, whilst the value of z corresponding to a length of 2.482 cm. is 1.774. The smallness of the variation of z implies that z , and therefore E , is unlikely to be affected to any serious extent by errors which would ordinarily be committed in determining quantities required for the evaluation of z .

The above discussion makes it clear that the main source of error in the determination of E is that due to non-uniformity of the thickness of the bar and the determination of this quantity. The probable effect of this error can be seen by comparing the values of E obtained, using two bars of the same material but of different thicknesses, *e.g.*, the results given above for the annealed specimens of nickel and monel; it is seen that in these cases, the discrepancy does not exceed 1 in 500 and we may safely conclude that the error in the above experimental results is unlikely to exceed 1 in 500. Inspection of the agreement between the values of Z , obtained with the three lengths taken for each specimen leads to approximately the same estimate for the accuracy of the results.

In proceeding to compare the above results with those
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of previous investigators, it must be remembered that Young's Modulus depends on the purity of the samples and also on their heat treatment. The above results show that the Young's Modulus of an annealed specimen is sometimes greater and sometimes smaller than that of the annealed specimen. It is interesting to note that (with the exception of Ferry and Glowray), when the Young's Modulus of the annealed specimen is greater than that of the unannealed specimen, then the resistivity of the annealed specimen exceeds that of the unannealed specimen, and *vice versa* (*v. table in Section VI.*)

It has been found impossible to trace any previous work on the values of the Young's Modulus of the above substances with the exception of nickel, nickel-iron alloys, and monel.

The results of previous determinations of the Young's Modulus of nickel are summarized in the table on p. 395.

In the case of nickel-iron alloys, it is possible to interpolate in order to calculate the Young's Modulus of 48 per cent. nickel-iron, since there appears^{(17), (5), (6)} to be a linear relationship between Young's Modulus and nickel content for nickel concentrations in the neighbourhood of 50 per cent. nickel. The values of Young's Modulus for various nickel-iron alloys have been determined by Honda⁽¹⁷⁾, Honda and Tanaka⁽⁵⁾, and Nakamura⁽⁶⁾; in each instance, the determination was carried out by a dynamical method. In the work of Honda⁽¹⁷⁾, the alloys had been annealed at 900° C., and interpolation from his results gives 15.99×10^{11} dynes/sq. cm. as the Young's Modulus of 48 per cent. nickel-iron. In the joint work of Honda and Tanaka⁽⁵⁾, the alloys were cast and then annealed for two hours at 900° C.; their results lead to a value of 16.01×10^{11} dynes/sq. cm. for the Young's Modulus of 48 per cent. nickel-iron. Interpolation from the results given by Nakamura⁽⁶⁾ leads to 15.09×10^{11} dynes/sq. cm. for the Young's Modulus of an alloy of the same composition. In the present work, the values obtained have been 13.65×10^{11} dynes/sq. cm. for the alloy annealed at 900° C. for 7 hours, and 15.29×10^{11} dynes/sq. cm. for the unannealed alloy.

In the case of monel, the Young's Modulus lies between 16.5 and 18.0×10^{11} dynes/sq. cm. according to the International Critical Tables, as compared with 14.94×10^{11} dynes/sq. cm. for the annealed alloy and 15.57×10^{11}

The Young's Modulus of Nickel.

Author.	Date.	D. or S.	Purity.	Heat treatment.	$E \times 10^{-11}$ dynes/sq. cm.)
Schaefer (8)	1901	S.	" Chemically pure," 97 % Ni.	...	23.10
Grüneisen (9)	1907	S.	99.9 % Ni.	Hard drawn.	20.15
Kurnakow and Rapke (10)	1914	S.	" Good,"	A. at 700° C.	19.72
Harrison (11)	1915	S.	" Good,"	A. at 500° C.	21.6
Koch and Dieterle (12)	1922	D.	" Chemically pure,"	A. at 620° C.	17.08
I. C. T.	1925	..	" Commercial."	...	20.0-21.4
Honda and Tanaka (5)	1926	D.	99.7 % Ni.	A. at 900° C. (2 hours).	19.31
Jacquierod and Mügeli (13)	1931	S.	" Very pure,"	A. for 10 hours.	20.01
Giebe and Blechschmidt (14)	1931	D.	" Technically pure,"	Hard drawn.	22.59
Zacharias (15)	1933	D.	98.9 % Ni.	Medium hard.	20.63
Nakamura (6)	1935	D.	" 100 % Ni."	A. at 700° C.	19.08
Siegel and Quimby (16)	1936	D.	99.7 % Ni.	A. at 900° C.	19.86
Present Authors	1936	D.	99.2 % Ni.	A. at 1100° C.	20.77
				A. at 1100° C. (4 hours).	20.94
				A. at 850° C.	19.37
				Unannealed.	19.18

Note.—In Column 1, "I. C. T." denotes "International Critical Tables." In column 3, "S" denotes that the determination was by a statitical method; "D" by a dynamical method. In the fifth column, "A" is used as an abbreviation for "annealed,"

dynes/sq. cm. for the unannealed alloy in the present work.

The values given above are seen to differ greatly from one another for each of the three substances; these discrepancies may very well be due to differences in the purity of the samples and in their heat treatment. As regards the latter, the table for nickel gives ample evidence for the pronounced effect of heat treatment on Young's Modulus; the results of Giebe and Blechschmidt⁽¹⁴⁾ show that a change of about 18 per cent. in Young's Modulus can be brought about by suitable heat treatment, whilst the results of Zacharias⁽¹⁵⁾ show a change of about 5 per cent. in Young's Modulus due to alteration of the annealing conditions. It is thus apparent that it is of little use giving the Young's Modulus of a substance unless the composition and the heat treatment of the sample are carefully specified.

In conclusion, the authors desire to express their indebtedness to Vice-Principal Gwilym Owen for his unfailing interest and encouragement, to Messrs. L. E. Sulston and D. Owen for their help in connexion with the construction of the apparatus, to Messrs. Henry Wiggin, Ltd., for the gift of the alloys, and to the Department of Scientific and Industrial Research for a maintenance grant to one of the authors (I. H. T.).

Note added in proof, January 13th, 1937.—A more detailed analysis of the theory of transverse vibration of fixed-free bars shows that there is a small error in the value of z^4 , which is given by equations (1.2) and (1.3) and which is used in the calculation of Young's Modulus. This error results from the fact that, in deriving equation (1.2), the rotatory inertia of the load and of the bar and the shear in the bar are considered to be negligible; a full discussion of the error due to these causes will shortly be published in this Journal by one of the authors (R. M. D.). In the present experiments the uncertainty in the final values of Young's Modulus due to this cause is unlikely to exceed 0.2 to 0.25 per cent.

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June 1936.

XXXVIII. *The Structure of Polished Metal Surfaces.* By S. DOBINSKI, Ph.D., Assistant in the Physical Laboratory, Jagellonian University, Cracow, Poland *.

ALTHOUGH investigations on the nature of polish can be traced so far back as the XVIIth century ⁽¹⁾ this problem has attracted an especial interest in the last few years. The recent development of electron diffraction technique has provided for the first time an experimental method particularly suitable for deciding the old question : whether the effect of polishing can be attributed to mere levelling of the surface or if it is accompanied by some changes in the *physical* state of the material near the surface.

The early experiments of G. P. Thomson ⁽²⁾ showed that, whereas the diffraction of electrons on metallic surfaces usually gave rise to patterns corresponding to the crystalline structure of the metal in question, it was impossible to obtain such patterns when the surface had been polished. It seemed to indicate a lack of any crystalline arrangement in a polished surface.

An identical supposition had been put forward several years before by G. Beilby ⁽³⁾. Microscopical observations led him to conclude that polishing causes a complete breakdown of crystalline structure, forming some sort

* Communicated by Prof. A. O. Rankine.

of amorphous, glass-like layer on the surface. Beilby's hypothesis was, however, rather sharply criticised⁽⁴⁾ and it was not until it obtained support in the entirely different way mentioned above that its popularity was considerably enhanced.

It stimulated, on the other hand, extensive research on the diffraction of electrons on polished surfaces. In spite of the fact, however, that quite consistent results have been obtained there is still a pronounced divergence of opinion as to their interpretation.

French, who was the first to undertake a more detailed examination of patterns obtained by diffraction on polished metals⁽⁵⁾, was able to show that they consisted of diffuse haloes exactly like X-ray patterns for glasses and liquids. His results were confirmed by subsequent investigations of Darbyshire and Dixit⁽⁶⁾, Raether⁽⁷⁾, and Miwa⁽⁸⁾. Raether proved that the patterns did not depend on the manner of polishing. The surfaces which had been burnished, hammered, or polished with rouge all gave patterns of the same kind. All these results might be easily accounted for by the presence of an amorphous layer on the polished surface.

There are, however, other factors which may act in a similar sense. We owe it to Kirchner⁽⁹⁾,⁽¹⁰⁾,⁽¹¹⁾, as well as Germer⁽¹²⁾,⁽¹³⁾, that their rôle has been fully emphasized.

It is generally accepted that patterns formed by so-called "reflexion" of electrons on different surfaces are actually due to transmission through small lumps and ridges projecting above the surface⁽¹⁴⁾. It is natural to suppose that these projections are being, at least partially, destroyed in the course of polishing. If it be possible to carry the process of polishing so far as to reduce the roughness of the surface to a scale comparable with the atomic dimensions, one would expect considerable blurring of the pattern due to insufficient resolution by those extremely minute projections which have been left. Kirchner is inclined to explain the differences of polish patterns on these lines rather than to attribute it to an amorphous layer. He supports this opinion by experiments showing that gold films condensed *in vacuo* on a flat substrate give very blurred patterns by reflexion, although quite sharp patterns obtained by transmission indicate that the films are not amorphous.

It is a rather inconvenient coincidence that mere levelling of the surface to a sufficiently high degree may influence electron diffraction in a sense similar to that due to breakdown of crystalline arrangement. G. P. Thomson was able to show^{(15), (16)}, however, that both effects are similar to the first approximation only. He pointed out that in the case of diffraction at a surface, of a degree of smoothness postulated by Kirchner, another factor would have become conspicuous, namely, that of inner potential. In this case the haloes would have departed from circular shape and their radii would have been much smaller than actually observed.

It is to be admitted, however, that, as far as the size of the haloes is concerned, the amorphous layer hypothesis also encounters considerable difficulties. From among the elements investigated all those of metallic character, with the single exception of tin⁽⁸⁾, have given haloes of practically the same size *. This uniformity is rather unexpected and, as was pointed out by Germer⁽¹³⁾, has no counterpart in X-ray diffraction by liquids. The size of the haloes should be governed by the atomic volume. It is true that the atoms of those metals which have been investigated are not supposed to show great differences in size, and, moreover, the diffuseness of the haloes does not allow accurate measurement of their radii. The comparison of results obtained by different authors shows, however, that an accuracy of about 5 per cent. is attainable while the atomic radii in question differ by more than 30 per cent.

There were some hypotheses proposed to account for this puzzling uniformity of the haloes, but the evidence in their favour could hardly be considered as convincing. Darbyshire and Dixit suppose that larger atoms, for which the discrepancies are the most serious, collapse in the course of polishing until their volume approaches that of the smallest metal atoms. So great a contraction under such a comparatively small pressure has, however, never been observed. Miwa tries to bridge the difficulty by assuming that, in cases which do not conform with the theory, the polish layer is not exactly amorphous, but consists of very small crystals with lattice constants different from those of the material in bulk.

* Haloes of different size have been obtained for the non-metallic elements Se and Si.

These or other rather artificial hypotheses were necessary to bring the amorphous layer theory in quantitative agreement with experimental results. Thus, although the experiments of Finch, Quarrell, and Roebuck⁽¹⁷⁾ on the growing of crystals on polished substrates have furnished further evidence that Beilby's hypothesis is much more constructive as a leading idea on the nature of polish than a purely geometrical explanation, objections against considering it as sufficiently founded have again been raised recently^{(11), (13)}, and still other speculations on the origin of diffuse rings have been tried.

So Germer hints at the possible effect of a layer of air or water absorbed on a polished surface. This would account for the uniform size of the rings, but, apart from other difficulties, one had to look for another explanation in those few cases where different radii of the haloes were reported.

In summarising it may be said that, although the amorphous layer theory had more evidence in its favour than any of the rival hypotheses, none of them fitted the quantitative results so well as to make efforts at explaining the phenomenon on still other lines unattractive. There was, however, the alternative possibility that the *experimental* data were in error, and it has been the purpose of the present work to elucidate this question.

The Preparation of Surfaces and Experimental Results.

The possibility of experimental errors being responsible for the difficulties might seem rather improbable, as the results obtained independently in different laboratories showed very good agreement between each other. There was very little hope of obtaining essentially different data by repeating the experiments following the same procedure. Closer consideration of the experimental methods employed showed, however, that there was one factor to which none of the investigators in question paid sufficient attention, namely, the possibility of oxidation in the course of polishing.

All the specimens examined until now were polished in air. The majority of the metals are known to be easily oxidizable, and even in those cases where the rate of oxidation is insignificant under normal conditions, it is

more often due to the presence of protective oxide layer than to the lack of chemical affinity. In the course of polishing this oxide layer is being continuously broken, and it does not afford, therefore, any protection. If polishing forms an amorphous layer on the surface still greater chemical activity might be expected, as the atoms which have been torn out from the crystalline array are the less likely to withstand chemical attack. If it is added that rouge or other oxides are generally used as polishing media, usually in the presence of water, it must be admitted that, by preparation of surfaces in this way, conditions most favourable for intensive oxidation are created.

It was clear that any attempt at revision of experimental data must first of all take this factor into account. It was decided, therefore, to investigate to what extent taking every possible precaution against oxidation might influence the results.

The most correct way would be, without doubt, to perform the entire process of polishing *in vacuo* in the camera itself. As it would have required a rather elaborate technique a much simpler procedure was adopted, which proved entirely satisfactory, the polishing being performed under the surface of benzene. The specimens were then placed wet in the camera and pumping immediately started so as to reduce the time of contact with air to a minimum.

The specimens were abraded on Hubert's emery papers Nos. from 1 to 0000 and subsequently polished on a piece of chamois. The strips of emery paper, as well as the chamois, were on the bottom of a flat, shallow, porcelain tray, as used for developing of photographic plates. The tray was filled with benzene, so that the specimen was all the time completely immersed. Benzene was chosen as a liquid unlikely to attack metals. To make sure, however, that the results are not affected by the presence of any impurities in benzene, in a few cases pentane was used. The results were identical.

As no polishing media such as rouge was used, it was difficult to achieve in this way a high polish in the case of soft metals lead and tin, and to some extent in the case of gold. The above-mentioned specimens were subsequently burnished with a polishing steel. Both the burnishing of the specimens and the previous polishing

of the burnishing tool were performed under the surface of benzene. Although the lead surface acquired a lustre after this treatment, it usually was not flat enough to give a good pattern; it was, therefore, pressed afterwards in a vice against a flat polished silver surface.

The specimens were examined in a camera of the Thomson-Fraser type⁽¹⁸⁾. Diffraction patterns consisting of the well-known diffuse haloes were obtained. The size of the haloes, however, have been found in most

TABLE I.
(a , d_1 , and d_2 are given in Å.)

	Z.	a .	d_1 .	d_2 .	$0.8.a.$	$\frac{0.8.a-d}{d}$
Al	13	2.85 ₇	2.36	1.24	2.28	-3 %
Fe	26	2.47 ₇	1.97	1.16	1.98	+1 %
Ni	28	2.48 ₉	1.99	1.16	1.99	0
Cu	29	2.55	1.97	1.16	2.04	+3 %
Ag	47	2.88 ₄	2.24	1.25	2.31	+3 %
Cd	48	2.97	2.36	1.27	2.38	+1 %
Sn	50	2.80	2.24	1.27	2.24	0
Au	79	2.87 ₈	2.23	1.27	2.30	+3 %
Pb	82	3.4 ₈	2.92	1.56	2.79	-4 %
Bi	83	3.11	2.38	1.33	2.48	+4 %

cases to depart distinctly from the uniformity reported by other authors. The corresponding spacings $d = \frac{\lambda}{\sin 2\theta}$ are given in the fourth and fifth columns of Table I. (λ denotes the electron wave-length, θ the Bragg angle.)

The fact that the sizes of the haloes obtained are different from those given by specimens polished in air shows that in the latter case the composition of the layers participating in electron diffraction was different. The simplest conclusion is that they were oxidized, and it explains the failure of efforts towards bringing them in a

quantitative agreement with the theory. It is interesting to confront now the new values with the requirements of the theory and to investigate what conclusions as to the structure of polished surfaces may be drawn on their basis.

Comparison with the Theory.

It is much more difficult to draw any definite conclusions as to the distribution of atoms in the case of amorphous bodies than for crystalline substances, nevertheless, in some simple cases the general outline of the pattern may be predicted. Thus the simplest case of a monatomic liquid, treated as a close-packed arrangement of spherical atoms agitated by the thermal motion, has been worked out thoroughly by J. A. Prins (19), (20), (21), (22). He was able to compute the intensity curve and to show that its first maximum, corresponding to the first halo of the pattern, should correspond to a spacing

$$d=0.80.a, \quad \dots \dots \dots \quad (1)$$

where a is the atomic diameter. The previous treatment of the same problem, on still more simplified assumptions, by Ehrenfest (23) as well as Keesom and de Smedt (24), (25), gave a very similar result $d=0.814.a$.

As the metals are, in general, monatomic, and as an amorphous layer formed by the crushing of the crystalline arrangement should resemble some sort of close-packing array, conditions very similar to those treated by Prins may be expected on a polished metal surface. To decide whether this is the case it is necessary to compare the sizes of the observed haloes with those calculated according to Prins's theory from the atomic diameters.

The idea of "atomic diameter" as the smallest possible distance of approach of the atomic centres is very useful in many theoretical considerations ; nevertheless, its value can scarcely be regarded as an atomic constant. It shows not only considerable differences when estimated by different methods, but depends also on such factors as the state of aggregation or temperature. It is clear, therefore, that, for the sake of an adequate comparison, the atomic diameters should be derived not only from similar phenomena, but also the experiments in question should be performed under conditions as similar as possible.

From this point of view it is hardly justifiable to apply to the Beilby layer the equation derived by Keesom for liquids,

$$a = 1.33^3 \sqrt{\frac{A}{\rho}},$$

where ρ denotes the density, A the atomic weight.

As no direct information on the density of the hypothetical Beilby layer is available, the corresponding values for the crystalline state of the metals in question were used. This is incompatible, however, with the assumptions under which Keesom's relation has been derived and cannot serve even as an approximation. The error is the bigger the further the structure departs from the closest packing, *e. g.*, it may be shown that for diamond structure it would amount to 30 per cent. ⁽²²⁾.

The interatomic distances which may be expected with the greatest probability in the Beilby layer are the distances between the nearest neighbours in the crystal lattice of the given metal. The above-mentioned data, taken from Ewald's 'Strukturbericht' ⁽²⁶⁾, are recorded in the third column of Table I. The values of d_1 , calculated from them according to Prins's formula, are given in column six, while the seventh column shows the percentage differences between them and the experimental values.

Discussion of Errors.

In order to decide whether the small differences between the experimental and calculated values are within the limits of those errors which may be expected on account of inaccuracy of the experiments or of the theory, it is convenient to consider the chief sources of error separately.

(a) Errors of Measurement.

The discussion of this sort of error is much simplified in this case, as the various usual sources of error in electron diffraction work are here quite insignificant in comparison with those which arise from the diffuseness of the pattern. It is difficult to estimate by mathematical analysis how far the latter affects the results, but some idea as to the accuracy may be obtained from comparison of the data given by different authors. The summary of the results for copper, which was one of the metals most frequently examined, are given as an example below. The last

column gives also values obtained by the present author *following the usual procedure of polishing in air*, and it will be seen that his results are *here* in agreement with the earlier ones.

It may be seen that the results do not differ among themselves by more than 5 per cent., although, being obtained on specimens polished in air, they all depart from the value for unoxidized copper (Table I.), as well as from that calculated from Prins's formula, by as much as 15 per cent. In conclusion, the value of about 5 per cent. may be accepted as the highest limit of error which may be attributed to the diffuseness of the haloes.

TABLE II.

French.			Darbyshire and Dixit.	Raether.	Miwa.	Dobinski.
d_1	2.26	2.33	2.28	2.24	2.28	2.24
d_2	1.22	1.30	1.26	1.25	1.29	1.28

(b) *Oxidation.*

Having demonstrated the fact that the ambiguity of the results, obtained until now, has been due in a much higher degree to oxidation than to inaccuracy of measurement, it is necessary to investigate whether and in how far the present values may be considered as unaffected by this factor. Although by preparation of the specimens in the manner described in the previous chapter, the surfaces were kept out of contact with air except for a short time before a sufficiently high vacuum in the camera had been reached, it was impossible to estimate in any other way than by trials whether or not these traces of oxidation, which might have taken place, did influence the results in a perceptible degree.

For this purpose polished specimens were exposed to air for different times and their diffraction patterns were compared. The rate of oxidation varied widely according to the kind of metal and the degree of humidity of the surrounding atmosphere. At any rate, even in the least favourable cases, it was not possible to detect any changes in diffraction patterns unless the time of exposure to air was of the order of a few hours. This proved that

the oxidation of the specimens in the camera, during the short time before it was exhausted, could not influence the results to any appreciable degree.

It is interesting to note that the oxidation occurred much sooner when the specimen had been exposed to air immediately after polishing than for a specimen preserved in benzene for some time after polishing.

If, however, the specimen had been exposed to air *during* polishing or burnishing, as previous experimenters have done, the oxidation was always complete, as far as the layers participating in electron diffraction are concerned. Any further exposure to air at room-temperature had no effect on the diffraction pattern. At higher temperatures sharp rings usually occurred indicating the formation of a crystalline form of the oxide. In cases of gold and silver only, the exposure to air either during or after polishing had no effect on the diffraction pattern.

(c) *Inaccuracy of the Theory.*

It is clear from the foregoing that errors of measurement alone may account for those small differences between the results and theoretical conclusions which are given in the seventh column of Table I. It may be shown, however, that, at least in some cases, a still better agreement might have been reached if the calculation had not been too simplified.

The Prins's formula may be written in the form

$$I = f^2 \phi(a, \theta),$$

where I is the intensity of the beam diffracted at an angle θ and f the atomic scattering power for electrons. The formula (1) (p. 403) gives, strictly speaking, the maximum of the function ϕ only, without allowing for the variation of f with θ . Taking the latter into account would result in a positive correction, generally a very

small one. The magnitude would depend on $\frac{1}{f} \frac{df}{d\theta}$. It is

well known that $\frac{1}{f} \frac{df}{d\theta}$ is greater for elements of lower atomic numbers and, for a given element, for smaller θ , *i. e.*, for bigger d .

The simplified formula (1) has actually given too low values for aluminium, the element of lowest atomic

number from among those investigated, and for lead with the greatest value of d . In both cases the above-mentioned correction would have brought them nearer to the experimental values.

Haloes produced by Microcrystalline Structures.

There might arise, perhaps, some doubts whether the data calculated on the assumption that the polished surface is not exactly amorphous, but composed of very minute, broken crystals, would not have given a still better agreement with the experimental results. It is easy to show, however, by taking as an example a metal of more complicated structure, that such a hypothesis is inapplicable. Every bismuth atom, for instance, has in the crystalline lattice three neighbours at the distance of 3.11 Å. and three 3.47 Å. apart. If the surface were composed of crystals, containing even a few structure units each, the distances 3.11 Å. and 3.47 Å. would occur with almost the same frequency, and each of them would give rise to a separate set of haloes.

Actually in case of bismuth specimens not carefully polished four haloes were observed. By further polishing, however, only the two corresponding to the distance 3.11 Å. survived.

This is incompatible with the assumption of the presence of very small crystals, and it is impossible also to account for by any theory which explains polish patterns as an effect of degeneration of the usual crystal patterns on account of refraction or insufficient resolution.

Conclusions.

The surface and the adjacent layers of a metal, polished in absence of air and other oxidizing media, consist of a close-packed ensemble of the corresponding atoms, irrespective of the normal crystalline structure of the given metal.

The distances of closest approach of the atoms in this amorphous layer are equal, within the limits of accuracy of the measurements, to those in the crystalline structure of the metal in question.

The surfaces polished in the commonly adopted manner, *i. e.*, in air, are heavily oxidized, especially when different oxides in the presence of water are used as polishing

media. The thickness of the oxidized layer produced in this way is much greater than that of a usual protective oxide layer formed by exposure of an unoxidized surface to air.

I should like to take the opportunity of expressing my sincere thanks to Professor G. P. Thomson for offering me the hospitality of his Laboratory and kind interest in this work, and to Professor A. O. Rankine for his kindness in providing me with every facility.

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XXXIX. *The Catalytic Combustion of Acetylene.*
*By W. DAVIES, Ph.D., B.Sc.**

THE experiments described in this paper form part of a general investigation of gaseous combustion in the heterogeneous phase on metallic surfaces, which is being carried out by a method ⁽¹⁾ based on observations of the temperatures attained by thin metallic wires when they are heated by an electric current and by catalytic combustion simultaneously in an atmosphere containing the reacting gases. For reasons which have been alluded to in a previous paper ⁽²⁾, a flame is not initiated at the surface of a thin filament in an inflammable gaseous mixture until the filament has been raised to a much higher temperature than the ignition point of the gas. It is therefore possible to study purely heterogeneous reactions by this method at high surface temperatures without complications due to the initiation of concurrent homogeneous reactions. This advantage is of special significance in connexion with this work, since it has been found that the surface temperatures required to promote the reaction of certain gases in the heterogeneous phase on platinum are higher than the mean gas temperatures at which the combustion of these gases can proceed in the gas phase.

The surface temperatures at which hydrogen, carbon-monoxide ^{(1), (2)}, methane ⁽³⁾, and acetylene begin to react with oxygen on platinum are of the order of 200, 400, 900, and 600° C. respectively, when experiments are carried out with small concentrations of each gas in air independently. This wide variation in the initiation temperatures for different reactions on the surface of the same catalyst is one of the necessary conditions for preferential combustion and the principle has long been applied in some well-known analytical methods based on the work of Henry ⁽⁴⁾, who showed that spongy platinum acts as a preferential catalyst for the oxidation of hydrogen and carbon-monoxide at temperatures where many of the hydrocarbons are unaffected. Compact platinum in the form of a wire also acts as a preferential catalyst in the same way under suitable conditions ⁽³⁾, but the experiments described in this paper show that a difference in the

* Communicated by Prof. W. T. David, Sc.D.

initiation temperatures for two reactions, when they are allowed to proceed independently, is not necessarily a criterion for preferential catalysis, since adsorption from a mixed atmosphere is generally a selective process and the conditions of reaction for one or the other of the gases may be modified to a considerable extent when they are both present. This effect is observed, for example, when acetylene is added to a mixture of hydrogen and air; the combustion of hydrogen, which is normally initiated at about 200° C., is now completely inhibited until the temperature of the wire is raised to the point at which the acetylene itself begins to react. The acetylene thus acts as an inhibitor or catalytic "poison" at temperatures below 600° C. and the wire is, therefore, unable to act as a preferential catalyst for the oxidation of hydrogen in these mixtures in spite of the fact that the two reactions are initiated independently at widely different temperatures. Methane, on the other hand, does not affect the conditions of reaction for hydrogen⁽³⁾ or acetylene and the wire acts preferentially towards either of these gases in the presence of methane, provided that its temperature is not raised to the point at which the methane itself begins to react.

The results in general lead to the conclusion that acetylene, unlike methane, is adsorbed by platinum at ordinary temperatures and that reaction occurs only when it begins to evaporate at a rate which allows oxygen to gain access to the surface.

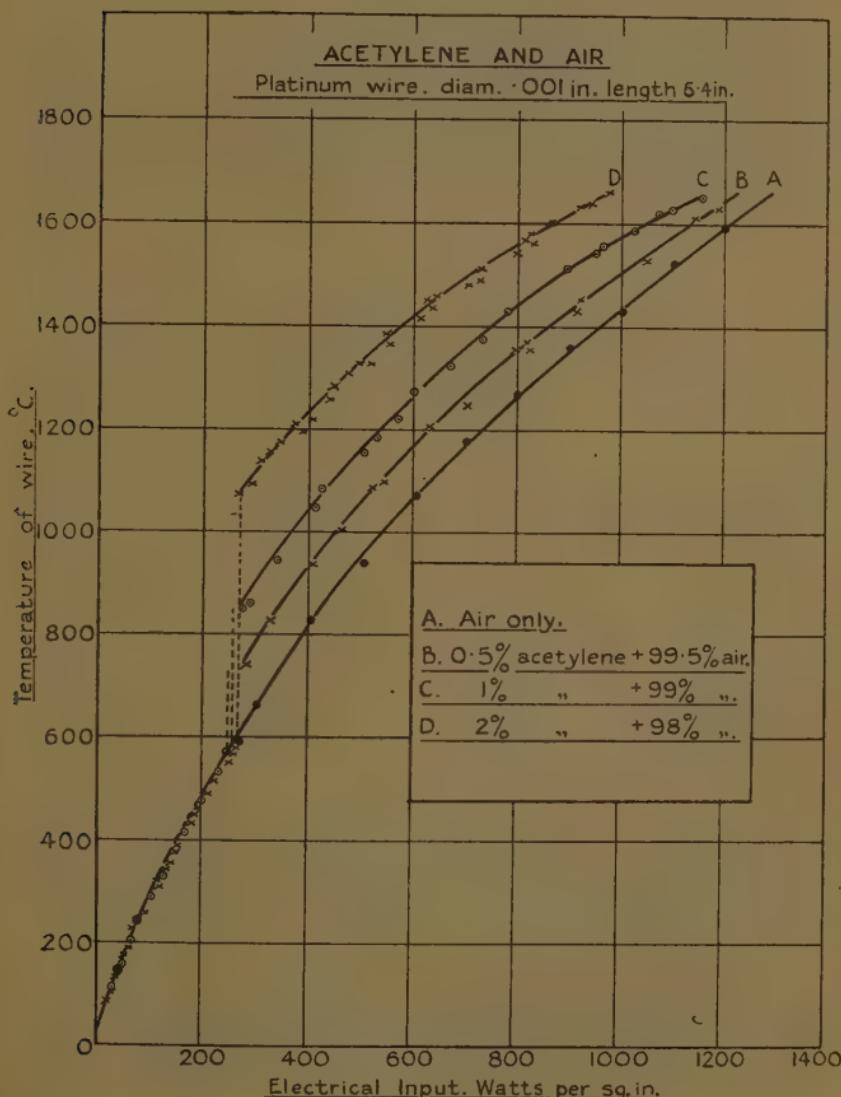
Results of Experiments.

(1) Acetylene and Air.

The results obtained with small concentrations of acetylene in air are shown in fig. 1. The curve A shows the temperatures attained by the wire when it is heated up in air by the current alone, and the curves B, C, and D show the temperatures attained by the same wire when it is heated by the current and by catalytic combustion simultaneously in mixtures of 0.5, 1 and 2 per cent. of acetylene and air, respectively. The wire shows no sign of activity in these mixtures until it has been heated by the current to about 600° C., since up to this point its temperature follows the air-curve A; reaction is then initiated more or less abruptly and the temperature now

rises suddenly from the level of A to the level of either B, C, or D, according to the amount of acetylene present in the mixture. The fact that these curves are nearly

Fig. I.



parallel to the air-curve shows that the rates of combustion are not greatly influenced by raising the temperature of the wire after reaction has been established; the actual values, as inferred from the horizontal intercepts

between the curves B, C, D, and the curve A, are given in Table I. These show that the rate of combustion at any given temperature is directly proportional to the concentration of the acetylene; therefore, it appears that the velocity of reaction in the heterogeneous phase is sufficiently high to enable the acetylene molecules to combine with oxygen as fast as they can diffuse to the active centres on the surface of the wire. The relatively small increase which takes place in the rate of combustion with any given concentration as the temperature of the wire rises up to about 1400° C. may be attributed to the

TABLE I.

Temperature of wire °C.	Rate of combustion. Watts per sq. in.		
	0.5	1	2
800	75	150	290
1000	85	165	330
1200	90	190	365
1400	100	200	390
1600	86	170	340

acceleration of diffusion and convection, but the falling off which occurs afterwards suggests that the active surface available for reaction is now diminishing, probably on account of the evaporation of oxygen which, according to Langmuir, takes place at an appreciable rate from platinum at temperatures above 1200° C. (5).

There is no evidence that combustion in the gas phase is taking place near the surface of the wire in any of these mixtures, even at temperatures above 1600° C., but when the concentration of acetylene is increased to just over 2.5 per cent. the mixture is ignited soon after the initiation of the heterogeneous reaction and a flame travels out slowly from the surface of the wire. The range of investigation with a wire of the given diameter must

therefore be restricted to mixtures containing less than this amount of acetylene.

The wires were suspended freely in a horizontal position in the gaseous mixtures, which were contained in a closed vessel of 1 cubic foot capacity; under these conditions the rate of combustion shown in Table I. were steadily maintained for a sufficient length of time to take the necessary readings before any sensible change due to the disappearance of acetylene by reaction could be detected. This steady condition being, of course, due to convection, which served to bring up a constant supply of fresh gas

TABLE II.

Combustible gas in air. (a).	Heat of combustion calories per gm. mol. (b).	Relative density. (c).	Rate of combustion. Watts per sq. in. at 1000° C. (d).	$\frac{(a).(b)}{\sqrt{(c).(d)}}$
1 per cent. H ₂	57,290	1	110	520
2 " "			228	502
1 per cent. CO	68,000	14	40	453
2 " "			83	440
1 per cent. C ₂ H ₂	312,000	13	165	525
2 " "			330	525
1 per cent. CH ₄	218,000	8	20	3855
2 " "			42	3680

to the wire and to carry away the products of combustion as fast as they were formed,

It is improbable that the rate of convection set up by the heated wire is influenced to any appreciable extent by the nature of the combustible gas when the concentrations are only of the order of 2 or 3 per cent., therefore the rate of heating of the wire by combustion at any given temperature should be directly proportional to the product of the concentration and the heat of combustion of the gas per gram molecule and inversely proportional to the square root of its density.

A comparison on this basis of the results obtained with different gases indicates, as shown in Table II., that an approximately equal fraction of the surface of the wire

is catalytically active for hydrogen, carbon-monoxide, and acetylene, whilst, at the same temperature only a comparatively very small area is active for methane. It should be stated, however, that the availability of the surface for the methane reaction increases rapidly and continuously with further rise of temperature up to the melting-point of platinum⁽³⁾, whilst above 1400° C. it begins to decrease for the other reactions^{(2), (3)}. The variation in the activity of the surface in these particular cases appears to be due to changes in the composition of the heterogeneous phase, but it is not improbable that the activity of the surface for some types of reaction may be dependent on structural alterations in the metal itself, which may occur with rise of temperature.

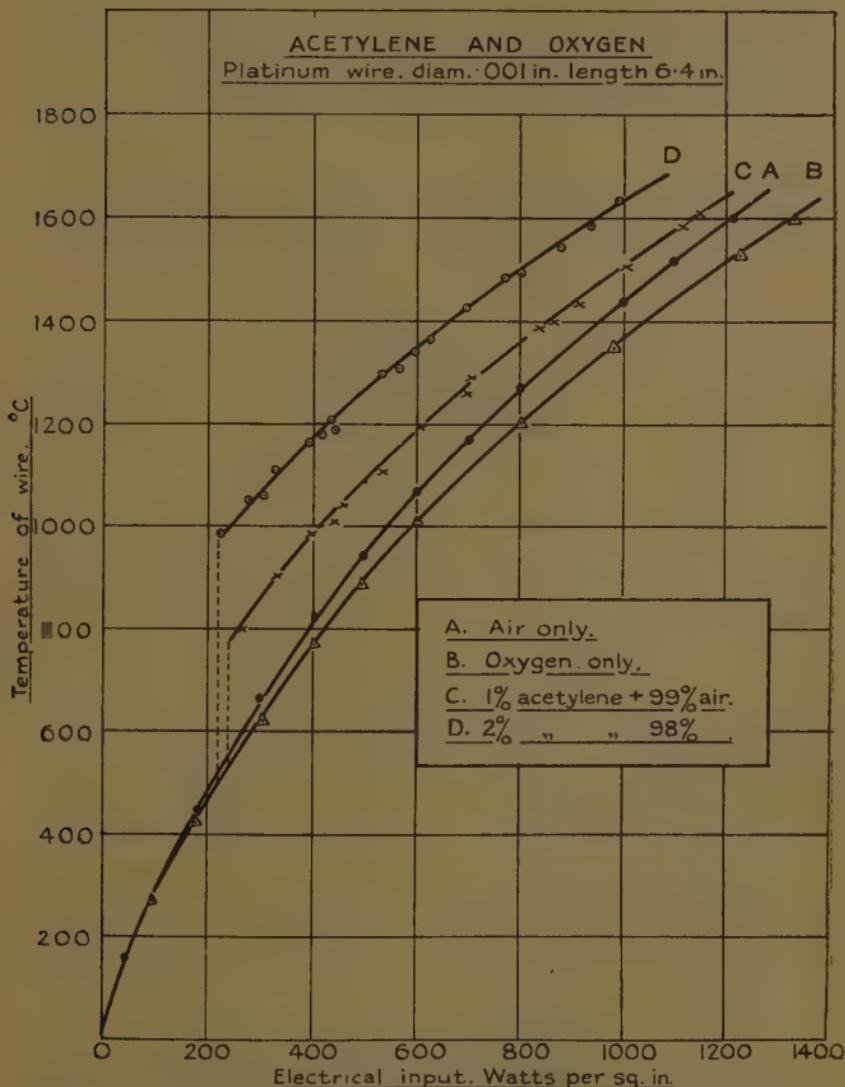
(2) *Acetylene and Oxygen.*

Nitrogen is not adsorbed by platinum to a measurable extent at any temperature⁽⁵⁾; therefore, in the experiments with the air mixtures described in the previous section, it merely acted as a diluent in the gas phase. The substitution of oxygen for air in these experiments, therefore, does not lead to any fundamental change in the composition of the adsorbed gases, but in virtue of the increase in the ratio of oxygen to acetylene in the gas phase there is now a greater probability that oxygen will reach the surface before the acetylene at the points where adsorbed atoms or molecules evaporate from time to time, and the net result is that oxygen gains access to the surface at a lower temperature than in the air mixtures. This view appears to be supported by the fact that the acetylene reaction is initiated at a somewhat lower temperature in oxygen than in air, but the subsequent rates of combustion are practically the same as in air. It may be recalled here that the substitution of oxygen for air in similar experiments with methane produces just the opposite effect in that reaction is delayed until the wire is heated to a considerably higher temperature than in air and the rate of combustion afterwards is retarded⁽⁸⁾.

The results obtained with 1 and 2 per cent. of acetylene in oxygen are shown in fig. 2; the curve B, which shows the temperatures attained by the wire when it is heated in oxygen by the current alone, is appreciably lower than

the air-curve A, especially at high temperatures; this is due to the fact that the wire is cooled more rapidly than in air, since the thermal conductivity of oxygen is greater

Fig. 2.

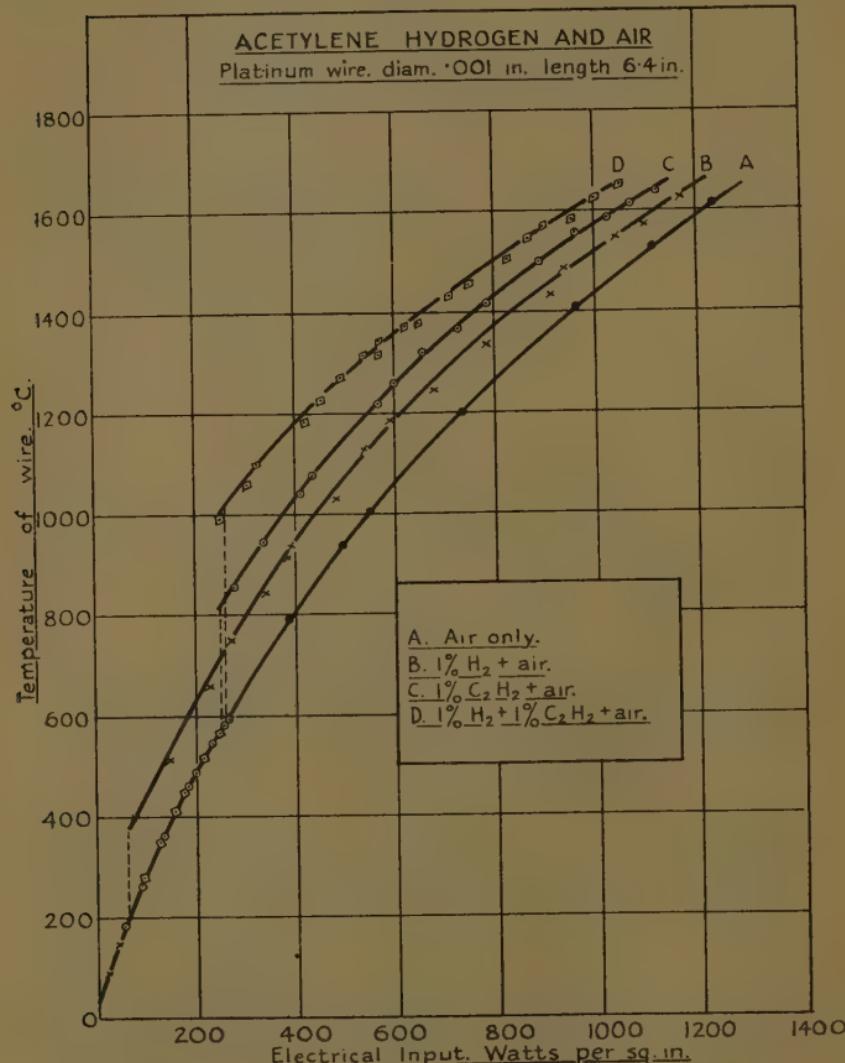


than that of nitrogen. The rates of combustion, which are now inferred from the horizontal intercepts between the oxygen curve B and the curves C and D are, within the limits of experimental error, equal to the corresponding values for the air mixtures given in Table I.

(3) *Acetylene, Hydrogen, and Air.*

In the presence of several reacting gases the behaviour of the wire will depend on the manner in which these

Fig. 3.



gases are selectively adsorbed by the metal. The results of experiments with different combinations of these gases should, therefore, throw some light on the conditions in the heterogeneous phase which promote or inhibit particular reactions. The curves B, C, and D in fig. 3

show the results obtained in a series of experiments with 1 per cent. of hydrogen, and an equal amount of acetylene in air. B shows the temperatures attained by the wire when the hydrogen only is present ; in this case reaction is initiated at about 200° C., and the temperature of the wire rises very suddenly at this point from the level of the air-curve A to the level of B, which is about 160° C. above A. The results obtained with the acetylene alone, shown by the curve C, have already been described in section (1). When both gases are present the hydrogen reaction is no longer initiated at 200° C., the temperature of the wire now follows the air-curve A up to nearly 600° C. before there is any indication of combustion whatsoever,

TABLE III.

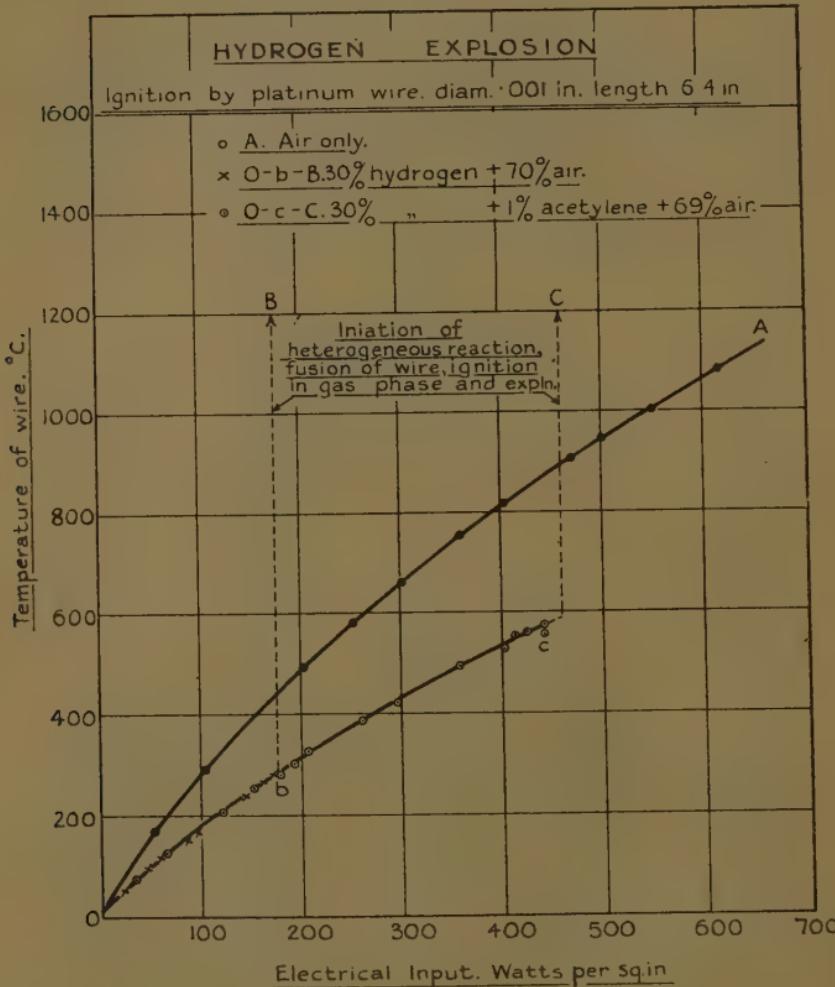
Temp. of wire °C.	Rate of combustion. Watts per sq. in.			
	1 % H ₂ (A).	1 % C ₂ H ₂ (B).	1 % H ₂ + 1 % C ₂ H ₂ .	(A)+(B).
1000	110	165	280	275
1200	125	190	320	315
1400	110	205	310	315
1600	80	170	245	250

but when reaction is initiated at this point the temperature rises to the level of D, which is about 160° C. above C. ; this indicates that the wire is now being heated by the acetylene and hydrogen reactions simultaneously. The rates of combustion, which are given in Table III., indicate moreover, that each reaction is contributing its full share to the heating effect.

The acetylene thus acts as an inhibitor, or catalytic poison, for the hydrogen reaction only at temperatures below 600° C. Its behaviour in this respect is characteristic of such " poisons " inasmuch that a very small concentration of it suffices to hold up the hydrogen reaction. This can be demonstrated most effectively in experiments with highly explosive mixtures of hydrogen and air, as shown in fig. 4. To explode such mixtures by means of a very fine platinum wire, it is only necessary to heat

the wire electrically to some point between 200 and 300° C.; the heterogeneous reaction which is then initiated raises the temperature of the wire almost instantaneously to its fusing point and ignition in the gas phase, followed by

Fig. 4.



explosion, occurs at some point before the wire actually melts. When, however, a very small amount of acetylene is added to the mixture this train of events is not started until the wire has been heated by the current to about 600° C. The results shown in fig. 4 were obtained in two experiments with 30 per cent. of hydrogen and air.

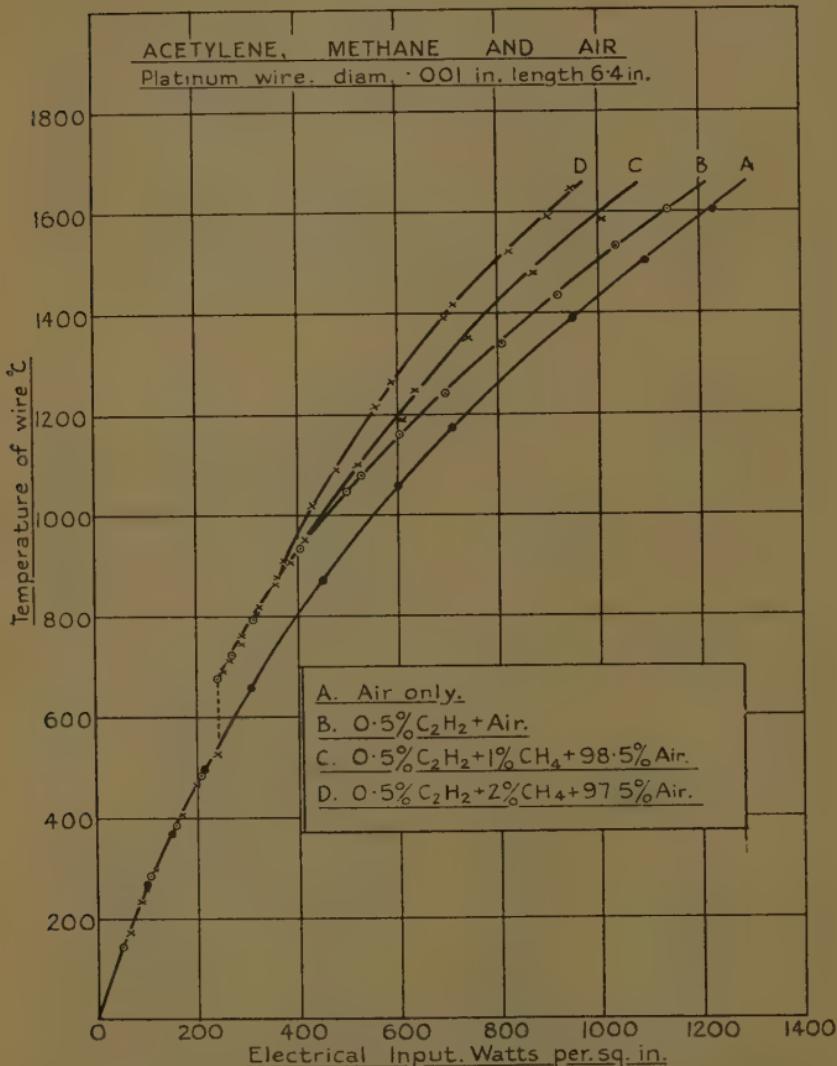
In the first, no acetylene was present and explosion occurred suddenly after the wire had reached a temperature of about 280° C., the explosion was, of course, preceded by an almost instantaneous rise of temperature in the wire due to the heterogeneous reaction, as indicated by the vertical dotted line *bB*. In the second experiment, 1 per cent. of acetylene was added to the mixture, and in this case explosion did not take place until after the wire had been heated by the current to nearly 600° C., as shown by *cC*. It will be observed that the points *b* and *c* are on a curve which lies far below the air-curve *A*; this is, of course, due to the cooling effect of the hydrogen which increases the thermal conductivity of the mixture very considerably when it is present in such large quantities.

(4) *Acetylene, Methane, and Air.*

In experiments with small quantities of acetylene and methane together in air there is no evidence of mutual interference and the wire acts as a preferential catalyst towards the acetylene in the interval which separates the initiation temperatures for the two reactions. In order to demonstrate this fact by this particular method it is obviously necessary to limit the amount of acetylene so that its own reaction does not raise the temperature of the wire to the point at which the methane begins to react. It is also necessary to limit the amount of methane, since the initiation temperature for the methane reaction decreases rapidly with increased concentration⁽³⁾ and the range of temperatures for the preferential combustion of acetylene is thus reduced. The experiments shown in fig. 5 were carried out with 0.5 per cent. of acetylene and up to 2 per cent. of methane. The results obtained with the acetylene alone, shown by the curve *B*, have already been described in section (1). When 1 per cent. of methane is added to this mixture the temperature of the wire still follows the acetylene curve up to about 950° C., then the methane begins to react and the temperature rises above the acetylene curve to *C*. When the concentration of methane is increased to 2 per cent. the methane reaction begins earlier, as shown by the curve *D*, and the range for the preferential combustion of the acetylene is thus less than it is with 1 per cent. The curves *C* and *D* diverge from the acetylene curve *B* in the same manner as they

would do from the air-curve A if the experiments were carried out with similar concentrations of methane alone

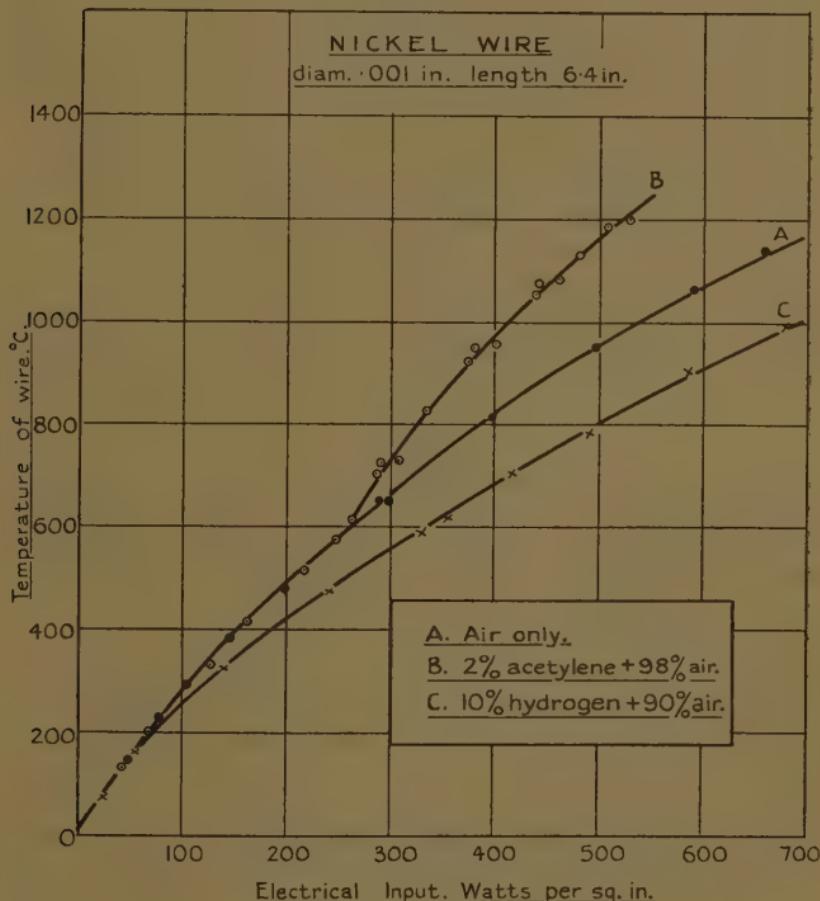
Fig. 5.



in air⁽³⁾. This divergence indicates that the rate of combustion of the methane increases continuously with rise of temperature up to the melting-point of platinum, whilst the rate of combustion of the acetylene decreases slightly with rise of temperature above 1400° C., as shown

in Table I. The methane and acetylene reactions thus preserve their individual characteristics when they occur simultaneously in these highly diluted mixtures, and it is inferred from this that they proceed quite independently at all temperatures.

Fig. 6.



(5) Metals other than Platinum.

With the single exception which is referred to below, all the tests that have hitherto been carried out on metals, other than those of the platinum group, by this method, have failed to give results which are in any way analogous to those obtained with platinum. Gold, silver, copper, and iron do not appear to promote catalytic combustion in the heterogeneous phase, but nickel proves an exception

in the present series in that it shows considerable activity as an oxidation catalyst for acetylene. The results shown by the curve B in fig. 6 were obtained with a nickel wire in a mixture of 2 per cent. of acetylene and air; reaction begins at approximately the same temperature as on platinum, but the subsequent rate of combustion is very much less than it is on a platinum wire of the same dimensions in this mixture. In view of the fact that finely divided nickel, when freshly reduced from its oxide, is capable of decomposing acetylene with evolution of heat⁽⁷⁾, it was thought that the above results might have been due to an action of this kind and not to an oxidation process, but experiments with similar concentrations of acetylene in nitrogen instead of in air fail to show any heating effect; therefore it is concluded that the wire is heated in the air mixture by catalytic combustion.

In order to show the contrast between the behaviour of nickel towards hydrogen and acetylene an experiment was carried out with a mixture of 10 per cent. of hydrogen and air, the result of which is shown by the curve C. In this case there is no indication of combustion up to 1100° C., and since the hydrogen now only adds to the cooling effect, the curve C stands well below the air-curve A. It should be stated, however, that soon after the wire has reached 1100° C. the mixture is ignited in the gas phase, but it has not yet been ascertained if ignition is preceded by the initiation of reaction in the heterogeneous phase, as in the experiments shown in fig. 4, since the temperature measurements become very uncertain in this region on account of the rapid change which takes place in the specific resistance of the nickel wire. Neither these wires nor the platinum wires were subjected to any form of preliminary chemical treatment, but they were first heated up to a high temperature in air to check their temperature coefficients, and before taking readings with the platinum wires they were also heated up to the maximum temperature in the mixtures to be used in the experiments, in order to ascertain by visual examination of the glowing surface that they were uniformly active from end to end.

Discussion.

The events leading up to the initiation of catalytic combustion on the wires in these experiments are clearly associated with the transitions which take place in the

heterogeneous phase with rise of surface temperature, and, to some extent, the nature of these transitions may be inferred from the results obtained with mixtures of different gases. It is obvious that reaction involves the adsorption of at least one of the reactants, and in the case of platinum, which appears to stand in a class of its own as an oxidation catalyst of general utility, the adsorption of oxygen seems to be an essential factor in all the reactions that have been studied up to the present. Langmuir⁽⁵⁾ has shown that this is in itself a sufficient condition for the catalytic oxidation of hydrogen and carbon-monoxide, and he found that hydrogen molecules from the gas phase will readily combine with adsorbed oxygen atoms on a platinum surface at all temperatures from the melting-point of platinum down to 15° C., or even lower. There is, on the other hand, no evidence that oxygen in the gas phase is able at any temperature to combine with adsorbed molecules or atoms of the combustible gases referred to in this paper. Langmuir⁽⁶⁾ states that carbon-monoxide and hydrogen, when adsorbed on platinum, act as catalytic "poisons" for any reactions which might otherwise take place if the surface were also accessible to other gases, and since carbon-monoxide and hydrogen are tenaciously adsorbed on platinum at ordinary temperatures, their catalytic oxidation in the heterogeneous phase becomes possible only when the temperature of the wire is above the points at which they evaporate at a sufficient rate to allow oxygen to gain access to the surface.

The results obtained with acetylene suggest that the initiation of reaction in this case also depends on a similar transition which takes place at about 600° C. The fact that the hydrogen reaction is inhibited at temperatures below this point when acetylene is present indicates that acetylene is adsorbed by platinum at ordinary temperatures, and since no reaction whatever takes place under these circumstances, it is evident that oxygen in the gas phase does not combine with adsorbed acetylene. These considerations lead to the conclusion that oxygen must reach the surface before reaction can proceed and this view is further supported by the fact that when combustion is eventually established at temperatures above 600° C. the conditions become equally favourable to the acetylene and hydrogen reactions.

The results obtained with mixtures of acetylene and

methane support the view put forward in a previous paper⁽³⁾ that methane is not adsorbed until the temperature of the wire reaches the point at which oxygen begins to evaporate and that reaction only proceeds when the methane and the oxygen are in the heterogeneous phase. With small concentrations of the order of 2 per cent. of methane in air this does not occur until the temperature is raised to about 900° C.; therefore, the methane should have no influence on the acetylene reaction, since it is not present in the heterogeneous phase until the temperature is far above the point at which the latter is initiated.

The behaviour of nickel towards acetylene is, so far, an isolated instance of its activity as an oxidation catalyst and this suggests that reaction in this case may be due to a temporary association between the acetylene and the metal. It seems certain that the mechanism of reaction is not the same as it is on platinum, since the rate of combustion is much less, although the wire appears to be uniformly active along the whole of its length when reaction is taking place. The discussion of these results must, however, be deferred until further experiments have been carried out with nickel wires in other hydro-carbon mixtures.

These experiments do not support the view that had become prevalent until recently that all solids tend to become equally effective as oxidation catalysts at elevated temperatures. This conclusion appears to have been reached on account of the failure of the usual experimental methods to distinguish between combustion in the heterogeneous phase and chain reactions or other types of reaction in the homogeneous phase.

Although the adsorption of oxygen appears to be the principal factor involved in bringing about the active state of platinum, it is not to be inferred that other metals which adsorb oxygen are necessarily active as oxidation catalysts. Langmuir⁽⁶⁾ has shown in some very striking experiments that when oxygen is adsorbed by different metals its activity varies enormously, and he found that oxygen adsorbed on tungsten is so inert that it will not combine with hydrogen even when the temperature of the surface is raised to over 1200° C.

My thanks are due to Professor W. T. David for his interest in this work, and to Mr. J. H. Marvell for valuable assistance in connexion with some of the experiments.

References.

- (1) Phil. Mag. xvii. p. 233 (1934).
- (2) Phil. Mag. xix. p. 309 (1935).
- (3) Phil. Mag. xxi. p. 513 (1936).
- (4) Phil. Mag. xlvi. p. 269 (1825).
- (5) Trans. Far. Soc. xvii. p. 624 (1922).
- (6) Phys. Rev. xiii. p. 188 (1933).
- (7) *Revue du Nickel*, pp. 18-20 (Jan. 1932).

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The University, Leeds.

XL. The Solution of a Set of Differential Equations which occur in the Theory of Saturn's Rings. By C. G. PENDSE, M.A., Ph.D. (of Downing College, Cambridge)*.

THE object of this paper is to give the details of the verification of a result, which was stated without proof in a previous paper † by the present writer, and to show, that though incidentally, the use of Fourier's theorem may not be permissible for obtaining the solution of a set of differential equations, the solution as actually obtained resembles a Fourier expansion, which in reality it is not.

In the above-mentioned paper the following $n(>1)$ interdependent pairs of linear differential equations of the second order with constant coefficients occur :—

$$\ddot{\rho}_\lambda - 2\omega \dot{\rho}_\lambda = \left[3\omega^2 + \frac{2m}{a^3} - \frac{m}{8a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1}{\sin \frac{n}{\pi\alpha}} + \frac{1}{\sin^3 \frac{n}{\pi\alpha}} \right\} \right] \rho_\lambda$$

* Communicated by Dr. W. M. Smart.

† C. G. Pendse, "The Theory of Saturn's Rings," Phil. Trans., A, ccxxxiv. pp. 145-176 (1935). The equations are given on p. 153. There is a misprint in the first line of the right-hand side of the equation (A. λ) on p. 153 of the paper. Read

$$\left[3\omega^2 + \frac{2m}{a^3} - \frac{m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1}{\sin \frac{\pi\alpha}{n}} + \frac{1}{\sin^3 \frac{\pi\alpha}{n}} \right\} \right] \rho_\lambda$$

instead of

$$\left[3\omega^2 + \frac{2m}{a^3} - \frac{m}{8a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1}{\sin \frac{\pi\alpha}{n} + \sin^3 \frac{\pi\alpha}{n}} \right\} \right] \rho_\lambda$$

on p. 153, line 11, of the Phil. Trans. paper.

$$\begin{aligned}
 & + \frac{m}{a^3} \sum'_{\mu} \left[\frac{1 + \sin^2 \frac{(\mu - \lambda)\pi}{n}}{8 \sin^3 \frac{(\mu - \lambda)\pi}{n}} \right. \\
 & \quad \left. + 2 \cos (\mu - \lambda) 2\pi/n \right] \rho_{\mu} \\
 & + \frac{m}{a^3} \sum'_{\mu} \left[\frac{1}{16 \sin^3 \frac{(\mu - \lambda)\pi}{n}} + 1 \right] \\
 & \quad \times \sin \frac{(\mu - \lambda) 2\pi}{n} \sigma_{\mu} \quad (A. \lambda) \\
 \ddot{\sigma}_{\lambda} + 2\omega \dot{\rho}_{\lambda} & = \frac{m}{a^3} \sum'_{\mu} \left[2 - \frac{1}{16 \sin^3 \frac{(\mu - \lambda)\pi}{n}} \right] \\
 & \quad \times \frac{\sin(\mu - \lambda) 2\pi}{n} \rho_{\mu} \\
 & + \frac{m}{a^3} \sigma_{\lambda} \sum_{\alpha=1}^{n-1} \left[\frac{1 + \cos^2 \frac{\pi \alpha}{n}}{8 \sin^3 \frac{\pi \alpha}{n}} + \cos \frac{2\pi \alpha}{n} \right] \\
 & - \frac{m}{a^3} \sum'_{\mu} \left[\frac{1 + \cos^2 \frac{(\mu - \lambda)\pi}{n}}{8 \sin^3 \frac{(\mu - \lambda)\pi}{n}} \right. \\
 & \quad \left. + \cos(\mu - \lambda) 2\pi/n \right] \sigma_{\mu}. \quad (B. \lambda)
 \end{aligned}$$

where $\lambda, \mu = 1, \dots, n$, and the summation sign Σ' means that the summation is carried out for μ , leaving out the value of μ for which it is equal to λ .

It will be useful to state the problem in connexion with which the equations (A. λ) and (B. λ) arise.

A collection of $n (> 1)$ particles of the same mass m is moving in a state of steady motion relative to the centre of a homogeneous spherical body. In the steady state the particles are assumed to form the vertices of a regular

polygon, inscribed in a circle radius a concentric with the central body and rotating with constant angular velocity ω . The system of particles is then disturbed in the plane of the circle, $a\rho_\lambda$ and σ_λ being the increments of the plane polar coordinates of the particle λ . Assuming that ρ_λ 's and σ_λ 's and their derivatives are small quantities of the first order and that quantities of higher orders of smallness are to be ignored, and that the system of particles and the sphere form an "isolated" system with the law of gravitation as the law of force between the members of the composite system, we obtain the pair of equations (A. λ) and (B. λ) for the disturbed motion of the particle λ . M is the mass of the central body and the gravitational unit of mass is chosen.

Equations similar to the equations (A. λ) and (B. λ) occur in J. Clerk Maxwell's Adams Prize Essay * on the subject. Tisserand † has followed Maxwell's method very closely. The present writer ‡ has shown in what respects his method of solving the n interdependent pairs of equations (A. λ) and (B. λ) differ from Maxwell's. A brief account of the method is given below.

The n pairs of equations (A. λ) and (B. λ) are interdependent, and it is necessary to derive from them n independent pairs of equations by a transformation of the dependent variables. Such a transformation is possible.

The transformation is given by

$$\rho_\lambda = \sum_{s=1}^n e^{\frac{2\pi i \lambda s}{n}} k_s, \quad \dots \dots \dots \quad (1/\lambda)$$

$$\sigma_\lambda = \sum_{s=1}^n e^{\frac{2\pi i \lambda s}{n}} l_s. \quad \dots \dots \dots \quad (2/\lambda)$$

The transformation is regular and its inverse is given by

$$k_s = \frac{1}{n} \sum_{\lambda=1}^n e^{-\frac{2\pi i \lambda s}{n}} \rho_\lambda, \quad \dots \dots \dots \quad (1'/s)$$

$$l_s = \frac{1}{n} \sum_{\lambda=1}^n e^{-\frac{2\pi i \lambda s}{n}} \sigma_\lambda. \quad \dots \dots \dots \quad (2'/s)$$

* "On the Stability of the Motion of Saturn's Rings," Scientific Papers, i. pp. 288-376. The relevant portion of the essay is §§ 1-7 of Part II.

† 'Traité de Mécanique Céleste,' t. ii. ch. xii.

‡ *Loc. cit.* § 4, pp. 159-160.

The following equations are obtained for the variables k_s and l_s :

$$\ddot{k}_s - 2\omega \dot{k}_s = [3\omega^2 + N + P_s]k_s + iQ_s l_s, \quad \dots \quad (A'.s)$$

$$\ddot{l}_s + 2\omega \dot{k}_s = iR_s k_s + T_s l_s, \quad \dots \quad (B'.s)$$

where

$$N = \frac{2m}{a^3} - \frac{m}{8a^3} \sum_{\alpha=1}^{n-1} \left[\frac{1}{\sin^3 \frac{\pi\alpha}{n}} + \frac{1}{\sin \frac{\pi\alpha}{n}} \right],$$

$$P_s = \frac{m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1}{8 \sin^3 \frac{\pi\alpha}{n}} + \frac{1}{8 \sin \frac{\pi\alpha}{n}} + 2 \cos \frac{2\pi\alpha}{n} \right\} \cos \frac{2\pi s\alpha}{n},$$

$$Q_s = \frac{m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1}{16 \sin^3 \frac{\pi\alpha}{n}} + 1 \right\} \sin \frac{2\pi\alpha}{n} \sin \frac{2\pi s\alpha}{n},$$

$$R_s = \frac{m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ 2 - \frac{1}{16 \sin^3 \frac{\pi\alpha}{n}} \right\} \sin \frac{2\pi\alpha}{n} \sin \frac{2\pi s\alpha}{n},$$

$$T_s = \frac{2m}{a^3} \sum_{\alpha=1}^{n-1} \left\{ \frac{1 + \cos^2 \frac{\pi\alpha}{n}}{8 \sin^3 \frac{\pi\alpha}{n}} + \cos \frac{2\pi\alpha}{n} \right\} \sin^2 \frac{\pi s\alpha}{n}.$$

Every one of the n pairs of linear differential equations (A'.s) and (B'.s), where $s=1, \dots, n$, is independent of the other pairs.

In the physical problem ρ_λ and σ_λ are real, while k_s and l_s will be complex (except when $s=n$).

It will be shown that the formulæ (1/λ) and (2/λ) lead to formulæ for ρ_λ and σ_λ , which are real functions of the independent variable t (time) and which involve $4n$ real arbitrary constants in all. This is the result to be proved in detail and was merely stated in the writer's previous paper *.

For k_n and l_n the equations (A'.s) and (B'.s) reduce to

$$\ddot{k}_n - 2\omega \dot{k}_n = 3\omega^2 k_n,$$

$$\ddot{l}_n + 2\omega \dot{k}_n = 0,$$

* Loc. cit. p. 160. The details concerning the derivation of the equations (A'.s) and (B'.s) and the formulæ (1'/s) and (2'/s) are given in § 3 of the former paper.

whose solutions are

$$k_n = 2a_n''' + a_n' \cos \omega t + b_n \sin \omega t$$

and

$$l_n = a_n''' - 3\omega a_n'' t + 2a_n'' \cos \omega t - 2a_n' \sin \omega t,$$

where a_n' , a_n'' , a_n''' , and a_n'''' are real arbitrary constants.

Considering a general pair (A'.s) and (B'.s) we assume, according to the usual method,

$$k_s = a_s e^{ip_s t}, \quad l_s = b_s e^{ip_s t}.$$

Then we have

$$a_s[(3\omega^2 + N + P_s) + p_s^2] = -ib_s[2\omega p_s + Q_s],$$

$$b_s[T_s + p_s^2] = ia_s[2\omega p_s - R_s].$$

Eliminating a_s and b_s ,

$$p_s^4 + (-\omega^2 + N + P_s + T_s)p_s^2 + 2\omega(R_s - Q_s)p_s + \{T_s(3\omega^2 + N + P_s) + Q_s R_s\} = 0. \quad (C.s.)$$

It is seen that, if $s < n$,

$$P_s = P_{n-s}, \quad Q_s = -Q_{n-s}, \quad R_s = -R_{n-s}, \quad T_s = T_{n-s}.$$

Hence, comparing equation (C.s) with the equation (C.n-s), we see that if x is a root of the equation (C.s) $-x$ is a root of the equation (C.n-s). Using this property and associating k_s with k_{n-s} , l_s with l_{n-s} ($s < n$) in the formulæ (1/λ) and (2/λ) for ρ_λ and σ_λ we shall obtain the formulæ for ρ_λ and σ_λ .

Let $p_{s,1}$, $p_{s,2}$, $p_{s,3}$, $p_{s,4}$ be the four roots of the equation (C.s). Then the solutions of the pair of differential equations (A'.s) and (B'.s) are

$$k_s = a_{s,1} e^{ip_{s,1} t} + a_{s,2} e^{ip_{s,2} t} + a_{s,3} e^{ip_{s,3} t} + a_{s,4} e^{ip_{s,4} t},$$

$$l_s = b_{s,1} e^{ip_{s,1} t} + b_{s,2} e^{ip_{s,2} t} + b_{s,3} e^{ip_{s,3} t} + b_{s,4} e^{ip_{s,4} t},$$

where $s < n$ and the arbitrary complex constants $a_{s,1}$ etc. and $b_{s,1}$ etc. are connected by the relations

$$a_{s,1} = -i \frac{[2\omega p_{s,1} + Q_s]}{3\omega^2 + N + P_s + p_{s,1}^2} b_{s,1},$$

$$a_{s,2} = -i \frac{[2\omega p_{s,2} + Q_s]}{3\omega^2 + N + P_s + p_{s,2}^2} b_{s,2},$$

$$a_{s,3} = -i \frac{[2\omega p_{s,3} + Q_s]}{3\omega^2 + N + P_s + p_{s,3}^2} b_{s,3},$$

$$a_{s,4} = -i \frac{[2\omega p_{s,4} + Q_s]}{3\omega^2 + K + P_s + p_{s,4}^2} b_{s,4}.$$

At this stage it is assumed that $s \neq \frac{1}{2}n$ (should n be even). Should n be even and $s = \frac{1}{2}n$, $n-s=s$ and k_{n-s} , l_{n-s} are identical with k_s , l_s , and consequently the pairing referred to above would not be possible in this case. A separate treatment is needed for $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$ (n even).

The sums of the contributions of k_s and k_{n-s} , and l_s and l_{n-s} to ρ_λ and σ_λ respectively are, using the formulæ (1/ λ) and (2/ λ),

$$e^{\frac{2\pi i \lambda s}{n}} k_s + e^{\frac{2\pi i \lambda (n-s)}{n}} k_{n-s}$$

and

$$\begin{aligned} & e^{\frac{2\pi i \lambda s}{n}} l_s + e^{\frac{2\pi i \lambda (n-s)}{n}} l_{n-s} \\ &= e^{\frac{2\pi i \lambda s}{n}} \{ b_{s,1} e^{ip_s, 1t} + b_{s,2} e^{ip_s, 2t} + b_{s,3} e^{ip_s, 3t} + b_{s,4} e^{ip_s, 4t} \} \\ & \quad + e^{\frac{2\pi i \lambda (n-s)}{n}} \{ b_{n-s,1} e^{ip_{n-s}, 1t} + b_{n-s,2} e^{ip_{n-s}, 2t} \\ & \quad \quad + b_{n-s,3} e^{ip_{n-s}, 3t} + b_{n-s,4} e^{ip_{n-s}, 4t} \} \\ &= e^{\frac{2\pi i \lambda s}{n}} \{ b_{s,1} e^{ip_s, 1t} + b_{s,2} e^{ip_s, 2t} + b_{s,3} e^{ip_s, 3t} + b_{s,4} e^{ip_s, 4t} \} \\ & \quad + e^{-\frac{2\pi i \lambda s}{n}} \{ b_{n-s,1} e^{-ip_s, 1t} + b_{n-s,2} e^{-ip_s, 2t} \\ & \quad \quad + b_{n-s,3} e^{-ip_s, 3t} + b_{n-s,4} e^{-ip_s, 4t} \} \end{aligned}$$

and

$$\begin{aligned} & e^{\frac{2\pi i \lambda s}{n}} k_s + e^{\frac{2\pi i \lambda (n-s)}{n}} k_{n-s} \\ &= e^{\frac{2\pi i \lambda s}{n}} \{ a_{s,1} e^{ip_s, 1t} + a_{s,2} e^{ip_s, 2t} \\ & \quad \quad + a_{s,3} e^{ip_s, 3t} + a_{s,4} e^{ip_s, 4t} \} \\ & \quad + e^{\frac{2\pi i \lambda (n-s)}{n}} \{ a_{n-s,1} e^{ip_{n-s}, 1t} + a_{n-s,2} e^{ip_{n-s}, 2t} \\ & \quad \quad + a_{n-s,3} e^{ip_{n-s}, 3t} + a_{n-s,4} e^{ip_{n-s}, 4t} \} \\ &= -ie^{\frac{2\pi i \lambda s}{n}} \left\{ \frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + P_{s,1}^2} b_{s,1} e^{ip_s, 1t} \right. \\ & \quad \quad \left. + \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + P_{s,2}^2} b_{s,2} e^{ip_s, 2t} \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} b_{s,3} e^{ip_{s,3}t} \\
 & \quad + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} b_{s,4} e^{ip_{s,4}t} \} \\
 & + ie^{\frac{2\pi i \lambda s}{n}} \left\{ \frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{n-s,1} e^{-ip_{s,1}t} \right. \\
 & \quad \left. + \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{n-s,2} e^{-ip_{s,2}t} \right. \\
 & + \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} b_{n-s,3} e^{-ip_{s,3}t} \\
 & \quad \left. + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} b_{n-s,4} e^{-ip_{s,4}t} \right\},
 \end{aligned}$$

using the relations between $a_{s,1}$ and $b_{s,1}$ etc., and remembering that $p_{s,1} = -p_{n-s,1}$ etc., and $Q_s = -Q_{n-s}$, $P_s = P_{n-s}$.

There are three distinct cases with respect to the roots of the equation (C.s) :—

(i.) All the roots of (C.s), and therefore those of (C. $n-s$), real and unequal.

(ii.) Either two or the four roots of (C.s), and therefore those of (C. $n-s$), complex and unequal.

(iii.) Two or more roots of (C.s), and therefore also those of (C. $n-s$), equal.

The case (iii.) might arise when all the roots are real. But it cannot arise when all the roots are complex ; for then the roots would have to be pure imaginary, of opposite signs and the same modulus, as the sum of the roots would have to be zero, and the equation would have to be of the form $(p_s^2 + \text{a positive number})^2 = 0$.

The slight modification due to case (iii.) will be mentioned when case (i.) is being developed.

(i.) All roots of the equation (C.s) real and unequal. We have

$$\begin{aligned}
 & l_s e^{\frac{2\pi i \lambda s}{n}} + l_{n-s} e^{-\frac{2\pi i \lambda s}{n}} \\
 & = (b_{s,1} + b_{n-s,1}) \cos \left(p_{s,1} t + \frac{2\pi \lambda s}{n} \right) \\
 & \quad + i(b_{s,1} - b_{n-s,1}) \sin \left(p_{s,1} t + \frac{2\pi \lambda s}{n} \right)
 \end{aligned}$$

$$\begin{aligned}
& + (b_{s,2} + b_{n-s,2}) \cos \left(p_{s,2} + \frac{2\pi\lambda s}{n} \right) \\
& + i(b_{s,2} - b_{n-s,2}) \sin \left(p_{s,2}t + \frac{2\pi\lambda s}{n} \right) \\
& + (b_{s,3} + b_{n-s,3}) \cos \left(p_{s,3}t + \frac{2\pi\lambda s}{n} \right) \\
& + i(b_{s,3} - b_{n-s,3}) \sin \left(p_{s,3}t + \frac{2\pi\lambda s}{n} \right) \\
& + (b_{s,4} + b_{n-s,4}) \cos \left(p_{s,4}t + \frac{2\pi\lambda s}{n} \right) \\
& + i(b_{s,4} - b_{n-s,4}) \sin \left(p_{s,4}t + \frac{2\pi\lambda s}{n} \right)
\end{aligned}$$

and

$$\begin{aligned}
& k_s e^{\frac{2\pi i \lambda s}{n}} + k_{n-s} e^{-\frac{2\pi i \lambda s}{n}} \\
& = \frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} \left[-i(b_{s,1} - b_{n-s,1}) \cos \left(p_{s,1}t + \frac{2\pi\lambda s}{n} \right) \right. \\
& \quad \left. + (b_{s,1} + b_{n-s,1}) \sin \left(p_{s,1}t + \frac{2\pi\lambda s}{n} \right) \right] \\
& + \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} \left[-i(b_{s,2} - b_{n-s,2}) \cos \left(p_{s,2}t + \frac{2\pi\lambda s}{n} \right) \right. \\
& \quad \left. + (b_{s,2} + b_{n-s,2}) \sin \left(p_{s,2}t + \frac{2\pi\lambda s}{n} \right) \right] \\
& + \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} \left[-i(b_{s,3} - b_{n-s,3}) \cos \left(p_{s,3}t + \frac{2\pi\lambda s}{n} \right) \right. \\
& \quad \left. + (b_{s,3} + b_{n-s,3}) \sin \left(p_{s,3}t + \frac{2\pi\lambda s}{n} \right) \right] \\
& + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} \left[-i(b_{s,4} - b_{n-s,4}) \cos \left(p_{s,4}t + \frac{2\pi\lambda s}{n} \right) \right. \\
& \quad \left. + (b_{s,4} + b_{n-s,4}) \sin \left(p_{s,4}t + \frac{2\pi\lambda s}{n} \right) \right].
\end{aligned}$$

Constraining the constants b 's in such a way that $b_{s,1}$ is the conjugate complex of $b_{n-s,1}$ etc., we see that the joint contributions of k_s , k_{n-s} to ρ_λ and l_s , l_{n-s} to σ_λ are real functions of t involving eight arbitrary constants.

The ordinary methods employed in the solution of differential equations can be applied to the joint contribu-

tions of k_s , k_{n-s} to ρ_λ and l_s , l_{n-s} to σ_λ as found above in case the equation (C.s) should have multiple roots. This is not likely to be necessary.

(ii.) The equation (C.s) has one set of conjugate complex roots.

Let the equation (C.s) possess a set of conjugate roots, say $p_{s,1}$ and $p_{s,2}$. Then $p_{s,1} + p_{s,2}$ is real and $p_{s,1} - p_{s,2}$ is pure imaginary. $p_{s,3}$ and $p_{s,4}$ are supposed to be real and unequal.

$$\begin{aligned}
 & l_s e^{\frac{2\pi i \lambda s}{n}} + l_{n-s} e^{-\frac{2\pi i \lambda s}{n}} \\
 &= (b_{s,3} + b_{n-s,3}) \cos\left(p_{s,3} t + \frac{2\pi \lambda s}{n}\right) \\
 &\quad + i(b_{s,3} - b_{n-s,3}) \sin\left(p_{s,3} t + \frac{2\pi \lambda s}{n}\right) \\
 &+ (b_{s,4} + b_{n-s,4}) \cos\left(p_{s,4} t + \frac{2\pi \lambda s}{n}\right) \\
 &\quad + i(b_{s,4} - b_{n-s,4}) \sin\left(p_{s,4} t + \frac{2\pi \lambda s}{n}\right) \\
 &+ e^{\frac{1}{2}i(p_{s,1} - p_{s,2})t} \left[b_{s,1} e^{\frac{1}{2}i(p_{s,1} + p_{s,2})t + \frac{2\pi i \lambda s}{n}} \right. \\
 &\quad \left. + b_{n-s,2} e^{-\frac{1}{2}i(p_{s,1} + p_{s,2})t - \frac{2\pi i \lambda s}{n}} \right] \\
 &+ e^{-\frac{1}{2}i(p_{s,1} - p_{s,2})t} \left[b_{s,2} e^{\frac{1}{2}i(p_{s,1} + p_{s,2})t + \frac{2\pi i \lambda s}{n}} \right. \\
 &\quad \left. + b_{n-s,1} e^{-\frac{1}{2}i(p_{s,1} + p_{s,2})t - \frac{2\pi i \lambda s}{n}} \right] \\
 &= (b_{s,3} + b_{n-s,3}) \cos\left(p_{s,3} t + \frac{2\pi \lambda s}{n}\right) \\
 &\quad + i(b_{s,3} - b_{n-s,3}) \sin\left(p_{s,3} t + \frac{2\pi \lambda s}{n}\right) \\
 &+ (b_{s,4} + b_{n-s,4}) \cos\left(p_{s,4} t + \frac{2\pi \lambda s}{n}\right) \\
 &\quad + i(b_{s,4} - b_{n-s,4}) \sin\left(p_{s,4} t + \frac{2\pi \lambda s}{n}\right) \\
 &+ e^{\frac{1}{2}i(p_{s,1} - p_{s,2})t} \left[(b_{s,1} + b_{n-s,2}) \cos\left(\frac{p_{s,1} + p_{s,2}}{2} t + \frac{2\pi \lambda s}{n}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
& + i(b_{s,1} - b_{n-s,2}) \sin\left(\frac{p_{s,1} + p_{s,2}}{2} t + \frac{2\pi\lambda s}{n}\right) \\
& + e^{\frac{1}{2}i(p_{s,2} - p_{s,1})t} \left[(b_{s,2} + b_{n-s,1}) \cos\left(\frac{p_{s,1} + p_{s,2}}{2} t + \frac{2\pi\lambda s}{n}\right) \right. \\
& \left. + i(b_{s,2} - b_{n-s,1}) \sin\left(\frac{p_{s,1} + p_{s,2}}{2} t + \frac{2\pi\lambda s}{n}\right) \right],
\end{aligned}$$

and

$$\begin{aligned}
& e^{\frac{2\pi i \lambda s}{n}} k_s + e^{-\frac{2\pi i \lambda s}{n}} k_{n-s} \\
& = \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} \left[-i(b_{s,3} - b_{n-s,3}) \cos\left(p_{s,3} t + \frac{2\pi\lambda s}{n}\right) \right. \\
& \left. + (b_{s,3} + b_{n-s,3}) \sin\left(p_{s,3} t + \frac{2\pi\lambda s}{n}\right) \right] \\
& + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} \left[-i(b_{s,4} - b_{n-s,4}) \cos\left(p_{s,4} t + \frac{2\pi\lambda s}{n}\right) \right. \\
& \left. + (b_{s,4} + b_{n-s,4}) \sin\left(p_{s,4} t + \frac{2\pi\lambda s}{n}\right) \right] \\
& - i \left[\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{s,1} e^{ip_{s,1}t + \frac{2\pi i \lambda s}{n}} \right. \\
& \left. - \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{n-s,2} e^{-ip_{s,2}t - \frac{2\pi i \lambda s}{n}} \right] \\
& - i \left[\frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{s,2} e^{ip_{s,2}t + \frac{2\pi i \lambda s}{n}} \right. \\
& \left. - \frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{n-s,1} e^{-ip_{s,1}t - \frac{2\pi i \lambda s}{n}} \right] \\
& = \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} \left[-i(b_{s,3} - b_{n-s,3}) \cos\left(p_{s,3} t + \frac{2\pi\lambda s}{n}\right) \right. \\
& \left. + (b_{s,3} + b_{n-s,3}) \sin\left(p_{s,3} t + \frac{2\pi\lambda s}{n}\right) \right] \\
& + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} \left[-i(b_{s,4} - b_{n-s,4}) \cos\left(p_{s,4} t + \frac{2\pi\lambda s}{n}\right) \right. \\
& \left. + (b_{s,4} + b_{n-s,4}) \sin\left(p_{s,4} t + \frac{2\pi\lambda s}{n}\right) \right]
\end{aligned}$$

$$\begin{aligned}
 & -ie^{\frac{1}{2}i(p_{s,1}-p_{s,2})t} \left[\frac{2\omega p_{s,1}+Q_s}{3\omega^2+N+P_s+p_{s,1}^2} b_{s,1} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t+\frac{2\pi i \lambda s}{n}} \right. \\
 & \quad \left. - \frac{2\omega p_{s,2}+Q_s}{3\omega^2+N+P_s+p_{s,2}^2} b_{n-s,2} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t-\frac{2\pi i \lambda s}{n}} \right] \\
 & -ie^{\frac{1}{2}i(p_{s,2}-p_{s,1})t} \left[\frac{2\omega p_{s,2}+Q_s}{3\omega^2+N+P_s+p_{s,2}^2} b_{s,2} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t+\frac{2\pi i \lambda s}{n}} \right. \\
 & \quad \left. - \frac{2\omega p_{s,1}+Q_s}{3\omega^2+N+P_s+p_{s,1}^2} b_{n-s,1} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t-\frac{2\pi i \lambda s}{n}} \right].
 \end{aligned}$$

Suppose, now, that the following pairs of constants are conjugate to one another :—

$$\begin{aligned}
 & b_{s,3} \text{ and } b_{n-s,3}; \quad b_{s,4} \text{ and } b_{n-s,4}; \quad b_{s,1} \text{ and } b_{n-s,2}; \\
 & b_{s,2} \text{ and } b_{n-s,1}.
 \end{aligned}$$

By hypothesis $p_{s,1}$ and $p_{s,2}$ are conjugate complex; therefore $i(p_{s,1}-p_{s,2})$ will be real. The sum of the contributions of b_s and b_{n-s} to the formula will then be a real function of t containing eight arbitrary real constants; for $b_{s,3}+b_{n-s,3}$, $b_{s,4}+b_{n-s,4}$, $i(b_{s,3}-b_{n-s,3})$, $i(b_{s,4}-b_{n-s,4})$, $b_{s,1}+b_{n-s,2}$, $b_{s,2}+b_{n-s,1}$, $i(b_{s,1}-b_{n-s,2})$, and $i(b_{s,2}-b_{n-s,1})$ will be real arbitrary constants when the b_s 's and b_{n-s} 's, which were originally arbitrary complex constants, have been restricted in the manner explained above.

It is easy to see that, with the same restrictions on the b 's, the sum of the contributions of k_s and k_{n-s} to the formula for ρ_λ will be a real function of t involving eight arbitrary real constants.

$$\begin{aligned}
 & -i(b_{s,3}-b_{n-s,3}) \cos\left(p_{s,3}t+\frac{2\pi\lambda s}{n}\right) \\
 & \quad + (b_{s,3}+b_{n-s,3}) \sin\left(p_{s,3}t+\frac{2\pi\lambda s}{n}\right)
 \end{aligned}$$

and

$$\begin{aligned}
 & -i(b_{s,4}-b_{n-s,4}) \cos\left(p_{s,4}t+\frac{2\pi\lambda s}{n}\right) \\
 & \quad + (b_{s,4}+b_{n-s,4}) \sin\left(p_{s,4}t+\frac{2\pi\lambda s}{n}\right)
 \end{aligned}$$

will be real functions of t .

$b_{s,1}$ is conjugate complex of $b_{n-s,2}$; therefore

$$b_{s,1} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t + \frac{2\pi i \lambda s}{n}}$$

is conjugate complex of

$$b_{n-s,2} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t - \frac{2\pi i \lambda s}{n}}.$$

Since $p_{s,1}$ and $p_{s,2}$ are conjugate complex, and since ω , N , P_s and Q_s are real, $2\omega p_{s,1} + Q_s$ is conjugate complex of $2\omega p_{s,2} + Q$ and $3\omega^2 + N + P_s + p_{s,1}^2$ is conjugate complex of $3\omega^2 + N + P_s + p_{s,2}^2$.

Therefore

$$\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{s,1} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t + \frac{2\pi i \lambda s}{n}}.$$

is conjugate complex of

$$\frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{n-s,2} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t - \frac{2\pi i \lambda s}{n}}.$$

For similar reasons

$$\frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{s,2} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t + \frac{2\pi i \lambda s}{n}}$$

is conjugate complex of

$$\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{n-s,1} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t - \frac{2\pi i \lambda s}{n}}.$$

Also $e^{\frac{1}{2}i(p_{s,1}-p_{s,2})t}$ is real, as $i(p_{s,1}-p_{s,2})$ is real.

Therefore

$$-ie^{\frac{1}{2}i(p_{s,1}-p_{s,2})t} \left[\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{s,1} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t + \frac{2\pi i \lambda s}{n}} \right. \\ \left. - \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{n-s,2} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t - \frac{2\pi i \lambda s}{n}} \right]$$

and

$$-ie^{\frac{1}{2}i(p_{s,2}-p_{s,1})t} \left[\frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} b_{s,2} e^{\frac{1}{2}i(p_{s,1}+p_{s,2})t + \frac{2\pi i \lambda s}{n}} \right. \\ \left. - \frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} b_{n-s,1} e^{-\frac{1}{2}i(p_{s,1}+p_{s,2})t - \frac{2\pi i \lambda s}{n}} \right]$$

will be real functions of t .

Hence $e^{\frac{2\pi i \lambda s}{n}} l_s + e^{-\frac{2\pi i \lambda s}{n}} l_{n-s}$ and $e^{\frac{2\pi i \lambda s}{n}} k_s + e^{-\frac{2\pi i \lambda s}{n}} k_{n-s}$

will be real functions of t involving eight arbitrary real constants when the b_s 's and b_{n-s} 's are so restricted as to make the former a real function of t .

If all the roots of the equation (C.s) are complex the method used above in connexion with one conjugate complex set can be applied to the other conjugate complex set. If $p_{s,3}$ and $p_{s,4}$ are real and equal the usual methods employed in the solution of differential equations can be applied to contributions due to $p_{s,3}$ and $p_{s,4}$ in the formulæ

for $e^{\frac{2\pi i \lambda s}{n}} l_s + e^{-\frac{2\pi i \lambda s}{n}} l_{n-s}$ and $e^{\frac{2\pi i \lambda s}{n}} k_s + e^{-\frac{2\pi i \lambda s}{n}} k_{n-s}$ obtained above.

To sum up,

the sums of the contributions of k_s, k_{n-s} to ρ_λ and of l_s, l_{n-s} to σ_λ ($s \neq \frac{1}{2}n$ if n be even), viz., $e^{\frac{2\pi i \lambda s}{n}} k_s + e^{-\frac{2\pi i \lambda s}{n}} k_{n-s}$ and $e^{\frac{2\pi i \lambda s}{n}} l_s + e^{-\frac{2\pi i \lambda s}{n}} l_{n-s}$ are real functions of t and involve eight arbitrary real constants.

The contributions of $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$ to the formulæ for ρ_λ and σ_λ , when n is even, will now be considered. From equations (1'/s) and (2'/s)

$$\left. \begin{aligned} k_{\frac{1}{2}n} &= \frac{1}{n} \sum_{\lambda=1}^n \rho_\lambda e^{-i\lambda\pi} = \frac{1}{n} \sum_{\lambda=1}^n \rho_\lambda \cos \lambda\pi, \\ l_{\frac{1}{2}n} &= \frac{1}{n} \sum_{\lambda=1}^n \sigma_\lambda e^{-i\lambda\pi} = \frac{1}{n} \sum_{\lambda=1}^n \sigma_\lambda \cos \lambda\pi, \end{aligned} \right\} (n \text{ even}).$$

Also $Q_{\frac{1}{2}n} = 0$, $R_{\frac{1}{2}n} = 0$. Hence all the coefficients in the pair of differential equations (A'. $\frac{1}{2}n$) and (B'. $\frac{1}{2}n$) are real. Further, $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$, as defined in terms of the ρ_λ 's and the σ_λ 's, are real. Without further analysis, therefore, it can be stated that $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$ are real functions of t involving four arbitrary real constants. If the roots of the equation are all real, $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$ will be trigonometric functions of t involving sines and cosines. If the equation (C. $\frac{1}{2}n$) has a pair of complex roots, or if all its roots are complex, there will be exponential terms in the formulæ for $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$.

The contributions of $k_{\frac{1}{2}n}$ and $l_{\frac{1}{2}n}$ to ρ_λ and σ_λ are $k_{\frac{1}{2}n} \cos \lambda\pi$ and $l_{\frac{1}{2}n} \cos \lambda\pi$, which are real and involve four arbitrary real constants.

The solutions of the n pairs of differential equations (A. λ) and (B. λ) when the roots of all the equations (C.s) are real and different can be written as follows :—

(I.) n odd :

$$\begin{aligned}
 \sigma_{\lambda} = & a_n''' - 2\omega a_n'' t + 2a_n'' \cos \omega t - 2a_n' \sin \omega t \\
 & + \sum_{s=1}^{\frac{1}{2}(n-1)} \left[(b_{s,1} + b_{n-s,1}) \cos \left(p_{s,1} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + i(b_{s,1} - b_{n-s,1}) \sin \left(p_{s,1} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + (b_{s,2} + b_{n-s,2}) \cos \left(p_{s,2} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + i(b_{s,2} - b_{n-s,2}) \sin \left(p_{s,2} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + (b_{s,3} + b_{n-s,3}) \cos \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + i(b_{s,3} - b_{n-s,3}) \sin \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + (b_{s,4} + b_{n-s,4}) \cos \left(p_{s,4} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + i(b_{s,4} - b_{n-s,4}) \sin \left(p_{s,4} t + \frac{2\pi\lambda s}{n} \right) \right],
 \end{aligned}$$

$$\rho_{\lambda} = 2a_n'' + a_n' \cos \omega t + a_n'' \sin \omega t$$

$$\begin{aligned}
 & + \sum_{s=1}^{\frac{1}{2}(n-1)} \left[\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} \left\{ -i(b_{s,1} - b_{n-s,1}) \cos \left(p_{s,1} t + \frac{2\pi\lambda s}{n} \right) \right. \right. \\
 & \quad \left. \left. + (b_{s,1} + b_{n-s,1}) \sin \left(p_{s,1} t + \frac{2\pi\lambda s}{n} \right) \right\} \right. \\
 & \quad \left. + \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} \left\{ -i(b_{s,2} - b_{n-s,2}) \cos \left(p_{s,2} t + \frac{2\pi\lambda s}{n} \right) \right. \right. \\
 & \quad \left. \left. + (b_{s,2} + b_{n-s,2}) \sin \left(p_{s,2} t + \frac{2\pi\lambda s}{n} \right) \right\} \right. \\
 & \quad \left. + \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} \left\{ -i(b_{s,3} - b_{n-s,3}) \cos \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right. \right. \\
 & \quad \left. \left. + (b_{s,3} + b_{n-s,3}) \sin \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right\} \right]
 \end{aligned}$$

$$+ \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} \left\{ -i(b_{s,4} - b_{n-s,4}) \cos\left(p_{s,4}t + \frac{2\pi\lambda s}{n}\right) + (b_{s,4} + b_{n-s,4}) \sin\left(p_{s,4}t + \frac{2\pi\lambda s}{n}\right) \right\} \Big],$$

$$(\lambda = 1, \dots, n),$$

where $a'_n, a''_n, a'''_n, a''''_n$ are arbitrary real constants and the b 's are arbitrary complex constants such that b_s is the conjugate complex of $b_{n-s,1}, \dots, b_{s,4}$ is the conjugate complex of $b_{n-s,4}$.

The total number of arbitrary real constants = $4 + 8 \times \frac{1}{2}(n-1) = 4n$.

(II.) n even :

$$\sigma_\lambda = a'''_n - 3\omega a''_n t + 2a''_n \cos \omega t - 2a'_n \sin \omega t + l_{\frac{1}{2}n} \cos \lambda \pi$$

$$+ \sum_{s=1}^{\frac{1}{2}n-1} \left[(b_{s,1} + b_{n-s,1}) \cos\left(p_{s,1}t + \frac{2\pi\lambda s}{n}\right) + i(b_{s,1} - b_{n-s,1}) \sin\left(p_{s,1}t + \frac{2\pi\lambda s}{n}\right) \right.$$

$$+ (b_{s,2} + b_{n-s,2}) \cos\left(p_{s,2}t + \frac{2\pi\lambda s}{n}\right) + i(b_{s,2} - b_{n-s,2}) \sin\left(p_{s,2}t + \frac{2\pi\lambda s}{n}\right)$$

$$+ (b_{s,3} + b_{n-s,3}) \cos\left(p_{s,3}t + \frac{2\pi\lambda s}{n}\right) + i(b_{s,3} - b_{n-s,3}) \sin\left(p_{s,3}t + \frac{2\pi\lambda s}{n}\right)$$

$$+ (b_{s,4} + b_{n-s,4}) \cos\left(p_{s,4}t + \frac{2\pi\lambda s}{n}\right) + i(b_{s,4} - b_{n-s,4}) \sin\left(p_{s,4}t + \frac{2\pi\lambda s}{n}\right) \Big].$$

$$\sigma_\lambda = 2a''_n + a'_n \cos \omega t + a''_n \sin \omega t + l_{\frac{1}{2}n} \cos \lambda \pi$$

$$+ \sum_{s=1}^{\frac{1}{2}n-1} \left[\frac{2\omega p_{s,1} + Q_s}{3\omega^2 + N + P_s + p_{s,1}^2} \left\{ -i(b_{s,1} - b_{n-s,1}) \cos\left(p_{s,1}t + \frac{2\pi\lambda s}{n}\right) + (b_{s,1} + b_{n-s,1}) \sin\left(p_{s,1}t + \frac{2\pi\lambda s}{n}\right) \right\} \right.$$

$$+ \frac{2\omega p_{s,2} + Q_s}{3\omega^2 + N + P_s + p_{s,2}^2} \left\{ -i(b_{s,2} - b_{n-s,2}) \cos\left(p_{s,2}t + \frac{2\pi\lambda s}{n}\right) \right.$$

$$\begin{aligned}
 & + (b_{s,2} + b_{n-s,2}) \sin \left(p_{s,2} t + \frac{2\pi\lambda s}{n} \right) \} \\
 & + \frac{2\omega p_{s,3} + Q_s}{3\omega^2 + N + P_s + p_{s,3}^2} \left\{ -i(b_{s,3} - b_{n-s,3}) \cos \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + (b_{s,3} + b_{n-s,3}) \sin \left(p_{s,3} t + \frac{2\pi\lambda s}{n} \right) \right\} \\
 & + \frac{2\omega p_{s,4} + Q_s}{3\omega^2 + N + P_s + p_{s,4}^2} \left\{ -i(b_{s,4} - b_{n-s,4}) \cos \left(p_{s,4} t + \frac{2\pi\lambda s}{n} \right) \right. \\
 & \quad \left. + (b_{s,4} + b_{n-s,4}) \sin \left(p_{s,4} t + \frac{2\pi\lambda s}{n} \right) \right\} , \\
 & \quad (\lambda = 1, \dots, n).
 \end{aligned}$$

where the arbitrary constants are subject to the same restrictions as in (I.). There are four arbitrary constants involved in the pair $k_{\frac{1}{2}n}, l_{\frac{1}{2}n}$; and there are eight arbitrary constants in each term in the summation. The total number of constants $= 4 + 4 + 8(\frac{1}{2}n - 1) = 4n$. If any one of the equations (C.s) has complex or multiple real roots, slight modifications, which have been explained above, will be necessary.

It is to be observed that part of the formulae for ρ_λ and σ_λ is a finite series of sines and cosines, but is not a Fourier series, though it looks like one. Finally, it cannot be assumed, *a priori*, that the disturbance can be split up into a finite number of independent disturbances of the Fourier type.

XLI. Note on Optical Resolution.

By A. BUXTON, M.A. Oxon.*

CONSIDER the reproduction of an infinitely distant self-luminous point-source such as a star, through an optical system such as a telescope. The image in the focal plane consists of a central concentration of light, surrounded by light rings of rapidly diminishing intensity. The intensity at a point near the symmetrical axis of the "spurious-disk" image can be conveniently represented in terms of Bessel Functions, thus :

$$I = \text{Intensity} = \left\{ \frac{2}{z} J_1(z) \right\}^2 \quad \text{where } z = \frac{2\pi Ry_1}{\lambda f}.$$

* Communicated by the Author.

R is the radius of the circular aperture, f is the focal length, λ the wave-length, and y_1 the distance of the point at which the intensity is considered from the centre of the "spurious disk." $J_1(z)$ is the Bessel Function of the first kind of order one.

The radii of the light rings are given by the values of y_1 when I has successive maxima and the radii of the dark rings are given by the zeros of I . The maxima occur where $J_2(z)=0$ and the first dark ring is given by the first zero of $J_1(z)$. This occurs when $z=3.83$.

It has been assumed by Lord Rayleigh and other writers that the eye can separate two point-sources if the centre of one image is on the circumference of the first dark ring of the other and *vice-versa*. Two infinitely distant point-sources can therefore be resolved where the angular separation is

$$\frac{3.83\lambda}{2\pi R} = \frac{1.2197 \lambda}{2R}.$$

It has been suggested that a less arbitrary criterion be adopted, viz., to the effect that the amplitude curves should intersect at their points of inflexion.

The points of inflexion occur where

$$\frac{d^2}{dz^2} \left(\frac{J_1(z)}{z} \right) = 0,$$

i. e., where

$$\frac{1}{z^2} J_2(z) - \frac{1}{z} J_2'(z) = 0;$$

$$\frac{J_2(z)}{z} = J_2'(z);$$

$$\frac{3J_2(z)}{z} = J_1(z).$$

As the tables of $J_0(z)$ and $J_1(z)$ are more detailed than the other functions, we may write this result in the form

$$J_1(z) = \frac{3}{z} \left(\frac{2}{z} J_1(z) - J_0(z) \right),$$

$$i. e., \quad J_1(z) \left\{ \frac{2}{z} - \frac{z}{3} \right\} = J_0(z).$$

From the tables we see that $z=2.30$ satisfies this equation, so that, adopting this criterion for the limits

of resolution, the angular separation is now $\frac{4.6\lambda}{2\pi R}$, which allows an increase of 20 per cent. over the previously accepted theoretical limit and approximately twice the limit due to a thin annular ring of the aperture instead of the full aperture. The best intensity curves, when out-of-focus, is combined with primary spherical aberration, bear a striking resemblance to the intensity curve for the Airy "spurious disk," so that here again presumably the theoretical limits would be increased by approximately the same percentage.

The fall in the intensity in the rings is very rapid, and Lord Rayleigh has shown that 90 per cent. of the whole light is received within the second dark ring. The outer rings can, therefore, exert little effect on the problem of the separation of point-sources.

Wider separation of two point-sources would give rise to the formation of a minimum at the mid-point on the resultant intensity curve, corresponding to the point of intersection of the two separate intensity curves.

In this case separation is more likely, whereas in the case when the proximity is so close that we obtain a maximum at this point, separation does not seem possible. In both the limits considered, there is a minimum at the middle of the resultant intensity curve, although in the second case the minimum is more pronounced, and so a higher contrast is obtained.

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XLII. Motions of Electrons in Magnetic Fields and Alternating Electric Fields. By L. G. H. HUXLEY, M.A., D.Phil., F. W. Bennett Lecturer in Physics, University College, Leicester *.

I. Introduction.

IT is shown in a previous paper ⁽¹⁾ that the velocity V_x of drift W_x of electrons through a gas in the direction of a steady electric field X , accompanied by a transverse magnetic field H along Oz , is

* Communicated by the Author.

$$W_x = \frac{2}{3} \frac{Xe}{m} \cdot \overline{\frac{T}{1+\omega^2 T^2} \left(1 + \frac{\omega^2 T^2}{1+\omega^2 T^2} \right)}. \quad \dots \quad (1)$$

The symbols in (1) have the following meanings :—

e is the electronic charge in e.m.u. and is considered positive ;

m the mass of the electron ;

$$\omega = \frac{He}{m} = H\rho ;$$

and T the mean time spent between collisions for those free paths traversed at velocity U , the velocity of agitation of the electron. Consequently, $T = l/U$ where l is the mean free path appropriate to the velocity U .

The bar surmounting expression (1) indicates that the average of the underlying quantity is to be taken with respect to T .

The expression for the associated velocity of drift W_y along Oy was also found. It is

$$W_y = - \frac{Xe}{m} \cdot \overline{\frac{\omega T^2}{(1+\omega^2 T^2)^2} \left(\frac{1}{3} + \omega^2 T^2 \right)}, \quad \dots \quad (2)$$

but for present applications it is convenient to transform this expression into the equivalent form

$$W_y = - \frac{2}{3} \frac{Xe}{m} \overline{\frac{\omega T^2}{(1+\omega^2 T^2)} \left(\frac{1}{2} + \frac{\omega^2 T^2}{1+\omega^2 T^2} \right)}. \quad \dots \quad (2a)$$

The deflexion θ of the stream of electrons is given by

$$\tan \theta = \frac{W_y}{W_x} = - \frac{\left(\frac{\omega T^2}{1+\omega^2 T^2} \right) \left(\frac{1}{2} + \frac{\omega^2 T^2}{1+\omega^2 T^2} \right)}{\left(\frac{T}{1+\omega^2 T^2} \right) \left(1 + \frac{\omega^2 T^2}{1+\omega^2 T^2} \right)}, \quad \dots \quad (3)$$

but an equivalent formula was discussed in the previous paper.

Except for the factor $2/3$, which is itself replaced by some other when the average is taken, formula (1) gives the same values for the velocity W_x as the formula hitherto employed when $\omega T \ll 1$, as usually occurs in laboratory experiments, but when $\omega T \gg 1$, as, for instance, within regions of interest in the ionosphere,

the values obtained for W_r are about double those calculated from the older formula.

In deriving expressions (1) and (2) care was taken to avoid an error of analysis inherent in many previous treatments of the motions of electrons, and to which Townsend ⁽²⁾ has drawn attention.

For the same reason, many of the formulæ ⁽³⁾ in current use for the conductivity of ionized gases in alternating electric fields, both with and without magnetic fields, are either incorrect in form or of restricted application. They appear to be principally derived indirectly from a theory of the propagation of electromagnetic waves through a dispersing medium, given by Lorentz ⁽⁴⁾, and, as they are of considerable importance in theories of the structure of the ionosphere, the correct formulæ are derived below by a direct method.

Before treating the motions in a magnetic field, we first investigate the velocity of drift W of electrons in a gas under the action of a simple harmonic electric field, in the absence of a magnetic field.

II. Instantaneous Velocity W in an Alternating Electric Field.

Consider a surface element dS at the origin O of a Cartesian system and lying in the plane xOy . Let dv be an element of volume $r^2 \sin \theta \, dr \, d\theta \, d\phi$ at a point $P(r, \theta, \phi)$, and suppose the region to be occupied by a gas throughout which electrons are uniformly distributed to the extent of n per unit volume. The number n is considered small compared with the number of gas molecules per unit volume, so that the only collisions of consequence are those between electrons and molecules.

An electron loses in general only a small proportion of its energy in a collision, and it will be supposed that it retains virtually the same velocity of agitation U for several successive collisions. It will first be supposed that the number n refers to those electrons whose velocities at any instant lie within the range dU about the velocity U . If l is the mean free path for electrons moving at velocity U , then the collision frequency $\nu = U/l$.

We shall investigate the effect of an electric field $Z = A \cos pt$ which acts along Oz , on the transport of electrons across the surface element dS .

It will be supposed that the amplitude A is restricted to small values, such that the velocities, communicated to the electrons over individual free paths, are small compared with the velocity of agitation U. Consequently, terms in the velocities involving the amplitude A to the second or higher degrees will, on this supposition, be rejected.

The number of electrons which leave dv within any solid angle $d\omega$ in time dt' is

$$nv dv dt' \frac{d\omega}{4\pi} \dots \dots \dots \quad (4)$$

Let θ' be the angle of incidence of electrons which arrive at dS from dv ; consequently, when Z is absent, $\theta'=\theta$. The normal section of the beam of electrons which traverse dS may be taken as $dS \cos \theta'$ *, and the corresponding solid angle

$$d\omega = \frac{dS \cos \theta'}{r^2} \dots \dots \dots \quad (5)$$

We shall obtain an expression for the angle θ' at time t when Z is $A \cos pt$.

We consider Z to act in its first quarter period in the direction $-Oz$. The velocity Δv_z , acquired in the field by an electron during its passage in time τ from dv to dS is, correct to terms of the first degree in A,

$$\Delta v_z = \rho A \int^{\tau} \cos p(t-\tau) d\tau = \frac{\rho A}{p} [\sin pt - \sin p(t-\tau)],$$

where

$$\rho = \frac{e}{m} \text{ and } \tau = \frac{r}{U}.$$

The component of the velocity U parallel to Oz on emission from dv , is

$$v_z = U \cos \theta,$$

and on arrival at dS ,

$$v_z + \Delta v_z = (U + \Delta U) \cos \theta',$$

whence

$$\cos \theta' = \cos \theta + \frac{\rho A}{Up} [\sin pt - \sin p(t-\tau)] \sin^2 \theta. \quad (6)$$

* *Vide* remarks at end of the paper.

We next determine the number of electrons which have originated in dv , and cross dS in the interval t to $t+dt$.

When an unchanging field Z operates, the time dt which elapses between the arrival at dS of two electrons from dv is the same as the interval dt' between their departure from dv ; when, however, Z varies with t , the intervals dt and dt' are no longer equal.

The relation between dt and dt' may be found as follows:—

Consider an electron which leaves dv at time t' in the direction PO. After time τ its velocity parallel to Oz is

$$\begin{aligned} v_z + A\rho \int_0^\tau \cos p(t'+\tau) d\tau \\ = v_z + \frac{A\rho}{p} [\sin p(t'+\tau) - \sin pt'] ; \\ = v_z + \Delta v_z . \end{aligned}$$

The orbital velocity $\frac{ds}{d\tau}$ at time $(t'+\tau)$ is therefore given by

$$\begin{aligned} \left(\frac{ds}{d\tau} \right)^2 &= v_x^2 + v_y^2 + (v_z + \Delta v_z)^2 ; \\ &\doteq U^2 + 2v_z \Delta v_z + . \end{aligned}$$

Thus, omitting the term in A^2 as small,

$$\left(\frac{ds}{d\tau} \right) \doteq U + \frac{v_z}{U} \Delta v_z = U + \cos \theta \Delta v_z .$$

The orbital distance s travelled in the time τ is, therefore,

$$\begin{aligned} s &\doteq U\tau + \cos \theta \int_0^\tau \Delta v_z ; \\ &= U\tau - \cos \theta \frac{A\rho}{p^2} [\cos p(t'+\tau) - \cos pt' + p\tau \sin pt'] . \end{aligned}$$

As the second term is small compared with the first, we obtain for the time τ to describe the distance s ,

$$\tau = \frac{s}{U} + \frac{\cos \theta}{U} \cdot \frac{A\rho}{p^2} \left[\cos p \left(t' + \frac{s}{U} \right) - \cos pt' + p \frac{s}{U} \sin pt' \right] .$$

The time taken to cover the same distance s by the electron which leaves dv at the subsequent time $(t'+\Delta t')$ is

$$\tau + \left(\frac{d\tau}{dt'} \right) \Delta t' = \tau + \frac{A\rho}{p} \cdot \frac{\cos \theta}{U} \left[p \frac{s}{U} \cos pt' + \sin pt' \right. \\ \left. - \sin p \left(t' + \frac{s}{U} \right) \right] \Delta t'.$$

Consequently, if $\Delta t'$ is the interval between the departure of two electrons from dv along PO, the interval Δt between their arrivals at dS is

$$\Delta t = \Delta t' + \frac{A\rho \cos \theta}{pU} \left[\frac{pr}{U} \cos pt' + \sin pt' \right. \\ \left. - \sin p \left(t' + \frac{r}{U} \right) \right] \Delta t'.$$

This formula may be transformed to give the interval $\Delta t'$ in terms of Δt as follows :

$$\Delta t' = \Delta t \left\{ 1 - \frac{A\rho \cos \theta}{pU} \left[\frac{pr}{U} \cos pt' + \sin pt' \right. \right. \\ \left. \left. - \sin p \left(t' + \frac{r}{U} \right) \right] \right\}. . . (7)$$

It follows from relations (4), (5), and (6) that the number df of electrons which collide within the element dv at P, and cross dS in the interval dt at time t , is

$$df = \frac{n \nu d\omega}{4\pi} dv dt' e^{-\tau/l},$$

where l is the mean free path.

Since $t' = t - r/U$, we obtain from (7)

$$dt' = dt \left\{ 1 - \frac{A\rho}{pU} \cos \theta \left[\frac{pr}{U} \cos p \left(t - \frac{r}{U} \right) \right. \right. \\ \left. \left. + \sin p \left(t - \frac{r}{U} \right) - \sin pt \right] \right\}.$$

On writing τ for r/U this becomes

$$dt' = dt \left\{ 1 - \frac{A\rho}{pU} \cos \theta [(p\tau \cos p\tau - \sin p\tau) \cos pt \right. \\ \left. - (1 - \cos p\tau - p\tau \sin p\tau) \sin pt] \right\}.$$

Consequently, when the term in A^2 is rejected, df is given by

$$df = \frac{dt dS}{4\pi} n \nu e^{-r/l} \sin \theta \left\{ \cos \theta \left[1 - \frac{A\rho}{pU} \cos \theta \right. \right. \\ \times \{(p\tau \cos p\tau - \sin p\tau) \cos pt \\ \left. \left. - (1 - \cos p\tau - p\tau \sin p\tau) \sin pt\} \right], \right. \\ \left. + \frac{\rho A}{pU} \sin^2 \theta [(1 - \cos p\tau) \sin pt + \sin p\tau \cos pt] \right\} dr d\theta d\phi.$$

In order to obtain the total number f^+ of electrons which traverse dS from the whole space above the plane xOy in time dt at time t , we write

$$\frac{r}{l} = \frac{r}{U} \cdot \frac{U}{l} = \frac{\tau}{T} = \nu\tau,$$

and integrate the above expression with respect, first to ϕ between the limits 0 to 2π , then to θ from 0 to $\pi/2$, and finally to τ from 0 to ∞ . Certain definite integrals are given for convenience in Appendix A.

We find

$$f^+ = \frac{n dt dS}{2} \left\{ \frac{U}{2} + \frac{2}{3} \frac{\rho AT}{(1+p^2T^2)} \left[\left(1 + \frac{p^2T^2}{1+p^2T^2} \right) \cos pt \right. \right. \\ \left. \left. + pT \left(\frac{1}{2} + \frac{p^2T^2}{1+p^2T^2} \right) \sin pt \right] \right\}, \\ = \frac{n dt dS}{2} \left\{ \frac{U}{2} + W_v \right\},$$

where W_v is the second term in the {}.

In the same way we find for the number f^- of electrons transported across dS in the reverse direction in time dt at time t ,

$$f^- = \frac{n dt dS}{2} \left\{ \frac{U}{2} - W_v \right\}.$$

Consequently, the net transport in time dt at time t across dS in the direction of the field $Z = A \cos pt$, is $f = f^+ - f^-$: that is,

$$f = n dt dS W_v.$$

Thus W_u is the instantaneous value of the velocity of drift in the field, of electrons animated with velocity of agitation U . On averaging over all velocities U we obtain the mean instantaneous velocity of drift W , as

$$W = \frac{2}{3} \overline{\frac{\rho AT}{(1+p^2T^2)}} \left(1 + \frac{p^2T^2}{1+p^2T^2} \right) \cos pt + \frac{2}{3} \overline{\frac{\rho AT}{(1+p^2T^2)}} \cdot pT \left(\frac{1}{2} + \frac{p^2T^2}{1+p^2T^2} \right) \sin pt. \quad . \quad (8)$$

According to this expression there exists between the oscillating current density neW and the applied field $A \cos pt$, a phase difference α given by

$$\tan \alpha = \left. \frac{-\left(\frac{pT^2}{1+p^2T^2} \right) \left(\frac{1}{2} + \frac{p^2T^2}{1+p^2T^2} \right)}{\left(\frac{T}{1+p^2T^2} \right) \left(1 + \frac{p^2T^2}{1+p^2T^2} \right)} \right\}. \quad . \quad . \quad (9)$$

Equations (8) and (9) embody several points of interest which deserve comment.

First, we note that for slowly varying fields such that $pT \ll 1$ the component of W in phase with the e.m.f. predominates, and the phase angle α , according to (9), approaches zero. The expression for W then approaches the form

$$W = \frac{2}{3} \rho A \bar{T} \cos pt,$$

and W is in phase with the e.m.f.

When $p \rightarrow 0$, the field assumes a steady value $Z = A$, so that W becomes

$$W = \frac{2Ze}{3m} \bar{T} = \frac{2Ze}{3m} \left(\frac{\bar{l}}{U} \right),$$

which is the correct formula for the velocity W in a constant field ⁽¹⁾, ⁽²⁾.

Second, we note, that when $pT \gg 1$, as occurs when the period of the e.m.f. is small compared with the mean interval between collisions of electrons with molecules, the expression for W approaches the limiting form

$$W = \frac{Ae}{pm} \sin pt.$$

This is the expression for the velocity of a free electron in an alternating field $Z = A \cdot \cos pt$. The current then lags 90° behind the e.m.f., as is also shown by (9).

The mean rate of dissipation of energy in collisions with the molecules is, per unit volume,

$$\left. \begin{aligned} \bar{w} &= \frac{p}{2\pi} \int_0^{2\pi/p} neWZ dt ; \\ &= \frac{2ne^2}{3m} \left[\left(\frac{T}{1+p^2T^2} \right) \left(1 + \frac{p^2T^2}{1+p^2T^2} \right) \right] \cdot \frac{A^2}{2} \end{aligned} \right\} \dots \quad (10)$$

Writing $\frac{A^2}{2} = (\bar{Z})^2$, where \bar{Z} is the r.m.s. value of the field, we observe that when $p \rightarrow 0$ (10) reduces to $neWZ$ the value for a steady e.m.f. (vide the following section for remarks on the limitations of equation (10)).

When $pT \gg 1$, we find

$$w \rightarrow \frac{4}{3} \frac{neZ^2}{mp^2} \left(\frac{1}{T} \right) = \frac{4}{3} ne \left(\frac{\bar{Z}}{p} \right)^2 \nu.$$

Finally, if equations (8) and (9) are compared with equations (1), (2 a), and (3), it will be seen that there is a striking similarity of form between the expression for the velocities W_x and W_y as functions of the quantity ω and the amplitudes of the components of the velocity W in the alternating field in terms of p ; and a similar resemblance in the expressions for the angles θ and α also. This suggests the method, given in the next section, for obtaining the expression for the velocity W in an alternating electric field accompanied by a transverse magnetic field.

As the velocity W is obtained as a linear function of the amplitude A , and lies in the direction of the field, we may superimpose other vibratory motions in directions normal to W without effect on W .

Consequently, a rotating electric field in the plane xOy produces a rotating velocity W of the same frequency whose representative vector lags behind that of the field by an angle α given by equation (9).

III. Motions in an Alternating Field accompanied by a Transverse Magnetic Field.

Let the electric field of amplitude A rotate at angular frequency p in the plane xOy , the magnetic field H being zero.

Then, as mentioned above, the velocity W also rotates at the same frequency, but lags behind the rotating field by an angle α given by equation (9).

The rotating components W_{\parallel} and W_{\perp} , of the velocity W , along and perpendicular to the rotating electric field, according to equation (8), are

$$W_{\parallel} = \frac{2}{3} \frac{\rho AT}{(1+p^2T^2)} \left(1 + \frac{p^2T^2}{1+p^2T^2} \right), \quad \left. \begin{aligned} W_{\perp} &= \frac{2}{3} \frac{pA\rho T^2}{(1+p^2T^2)} \left(\frac{1}{2} + \frac{p^2T^2}{1+p^2T^2} \right). \end{aligned} \right\} \dots \quad (11)$$

and

We wish to find the values of these components when a magnetic field H is established parallel to the positive direction Oz .

The following empirical procedure is suggested :—

Under the influence of the rotating electric field electrons in a small volume acquire a velocity of drift along a circular path. The instantaneous velocity W , on account of the centrifugal force $nWe \left(\frac{mp}{e} \right)$ per unit volume normal to the path, possesses components W_{\perp} and W_{\parallel} (equation 11) in the direction of and perpendicular to the electric force.

Reference to equations (1) and (2a) shows that the components W_{\parallel} and W_{\perp} are actually those appropriate to a stationary electric field accompanied by a magnetic field $H_p = \frac{mp}{e}$ normal to it.

We assume that when an actual field H in the direction Oz accompanies the electric field rotating in the plane xOy , then the components W_{\parallel} and W_{\perp} are those appropriate to a stationary electric field and a total effective magnetic field $(H + H_p) = \frac{m}{e}(\omega + p)$, where $\omega = \frac{He}{m}$.

Consequently, in this circumstance,

$$W_{\parallel} = \frac{2}{3} \rho A \left(\frac{T}{1+(p+\omega)^2T^2} \right) \left(1 + \frac{(p+\omega)^2T^2}{1+(p+\omega)^2T^2} \right), \quad \left. \begin{aligned} W_{\perp} &= \frac{2}{3} \rho A \left(\frac{(p+\omega)T^2}{1+(p+\omega)^2T^2} \right) \left(\frac{1}{2} + \frac{(p+\omega)^2T^2}{1+(p+\omega)^2T^2} \right). \end{aligned} \right\} \quad (12)$$

The projection of these rotating velocities on the direction Ox in the original stationary system gives a velocity along Ox varying harmonically with the time according to

$$W_{+p} = \frac{2}{3} \rho A \left(\frac{T}{1+(p+\omega)^2 T^2} \right) \left(1 + \frac{(p+\omega)^2 T^2}{1+(p+\omega)^2 T^2} \right) \cos pt + \frac{2}{3} \rho A \left(\frac{(p+\omega) T^2}{1+(p+\omega)^2 T^2} \right) \left(\frac{1}{2} + \frac{(p+\omega)^2 T^2}{1+(p+\omega)^2 T^2} \right) \sin pt. \quad \dots \quad (13)$$

To obtain the velocities W , which result from the application of a simple harmonic field $X = A \cos pt$ along Ox when a magnetic field acts along Oz , we suppose the field X resolved into two component fields each of amplitude $\frac{A}{2}$ and rotating at angular frequency $+p$ and $-p$ respectively, the effective magnetic field in the latter case being $\frac{m}{e}(p-\omega)$ in the direction $-Oz$.

Projection of the corresponding rotating components

$(W_{||})_{+p}$, $(W_{||})_{-p}$, $(W_{\perp})_{+p}$ and $(W_{\perp})_{-p}$ on Ox gives the following expression for the velocity W_x :

$$W_x = \frac{2}{3} \rho \left(\frac{A}{2} \right) T \left[\frac{1}{1+(\omega+p)^2 T^2} \cdot \left(1 + \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) + \frac{1}{1+(\omega-p)^2 T^2} \left(1 + \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right] \cos pt + \frac{2}{3} \rho \left(\frac{A}{2} \right) T^2 \left[\frac{(\omega+p)}{1+(\omega+p)^2 T^2} \left(\frac{1}{2} + \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) - \frac{(\omega-p)}{1+(\omega-p)^2 T^2} \cdot \left(\frac{1}{2} + \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right] \sin pt. \quad \dots \quad (14)$$

The velocity along Ox is accompanied by a velocity W_y along O_y , given by

$$W_y = \frac{2}{3} \rho \left(\frac{A}{2} \right) T \left[\frac{1}{1+(\omega+p)^2 T^2} \left(1 + \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) - \frac{1}{1+(\omega-p)^2 T^2} \left(1 + \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right] \sin pt$$

$$-\frac{2}{3}\rho \binom{A}{2} T^2 \left[\frac{(\omega+p)}{1+(\omega+p)^2 T^2} \left(\frac{1}{2} + \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) \right. \\ \left. + \frac{(\omega-p)}{1+(\omega-p)^2 T^2} \left(\frac{1}{2} + \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right] \cos pt. \quad (15)$$

Equations (14) and (15) include all the previous formulæ as special instances. For example, when $p=0$, we obtain equations (1) and (2a); when $\omega=0$, W_y vanishes and W_x is given by equation (8).

When both ω and p are zero, (14) becomes

$$W_x = \frac{2}{3} \rho A \bar{T},$$

the correct formula for the velocity under a steady force A.

The current density in the direction Ox is neW_r and the mean rate of expenditure energy $\bar{\omega}$, by the field is, per unit volume,

$$\frac{p}{2\pi} \int_0^{2\pi/p} neW_x X dt,$$

whence

$$\dot{w} = \frac{2}{3} \frac{ne^2}{m} \left(\frac{A^2}{2} \right) \cdot \overline{T} \left[\frac{1}{1 + (\omega + p)^2 T^2} \left(1 + \frac{(\omega + p)^2 T^2}{1 + (\omega + p)^2 T^2} \right) \right. \\ \left. + \frac{1}{1 + (\omega - p)^2 T^2} \left(1 + \frac{(\omega - p)^2 T^2}{1 + (\omega - p)^2 T^2} \right) \right]. \quad . \quad (16)$$

When, however, $pT \ll 1$, an electron makes a great many collisions during a cycle of the applied field and appreciable fluctuations will occur in the mean energy of agitation $\frac{1}{2}mU^2$.

The coefficients of $\cos pt$ and $\sin pt$ in equations (8) and (14) for the instantaneous velocities W then become periodic functions of the time, with fundamental period $p/2\pi$.

Expressions (10) and (16) for \bar{w} would in these circumstances be inaccurate.

However, as the mean fractional loss of energy in individual collisions is in general small, the error in \bar{w} is inappreciable unless $pT << 1$.

IV. Calculation of Mean Rate of Dissipation of Energy.

As a check on the preceding results a direct calculation is given below of the mean rate of dissipation of energy in collisions when the alternating electric field is accompanied by a magnetic field.

Without loss of generality the axis Oz may be directed parallel to the field H and the plane xOz chosen to contain the electric vector and H.

Let X and Z be the components of the electric force along Ox and Oz respectively, and suppose them to be simple harmonic functions of the time,

$$X = A \cos(pt + \beta),$$

$$Z = B \cos(pt + \gamma).$$

Consider an electron which suffers a collision at time $t=0$, at the origin, and proceeds to move with an initial velocity U whose components are v_x , v_y , and v_z .

The equations of motion are

$$\left. \begin{array}{l} \ddot{x} = \rho X + \omega \dot{y}, \\ \ddot{y} = -\omega \dot{x}, \\ \ddot{z} = \rho Z, \end{array} \right\} \dots \dots \dots \quad (17)$$

where $\rho = \frac{e}{m}$ and $\omega = H\rho$.

The solutions of these which satisfy the initial conditions, are

$$\left. \begin{array}{l} x = C \sin(\omega t + \alpha) - D p \sin(pt + \beta), \\ y = C \cos(\omega t + \alpha) - D \omega \cos(pt + \beta), \\ z = \frac{B}{p} [\sin(pt + \gamma) - \sin \gamma] + v_z. \end{array} \right\} \dots \dots \quad (18)$$

$$\left. \begin{array}{l} x = \frac{C}{\omega} [\cos \alpha - \cos(\omega t + \alpha)] + D [\cos(pt + \beta) - \cos \beta], \\ y = -\frac{C}{\omega} [\sin \alpha - \sin(\omega t + \alpha)] - \frac{D \omega}{p} [\sin(pt + \beta) - \sin \beta], \\ z = -\frac{B}{p^2} [\cos(pt + \gamma) - \cos \gamma] + [v_z - \frac{B}{p} \sin \gamma] t, \end{array} \right\} \dots \dots \quad (19)$$

where

$$\begin{aligned}
 D &= \frac{\rho A}{(\omega^2 - p^2)}, \\
 C \sin \alpha &= v_x + Dp \sin \beta, \\
 C \cos \alpha &= v_y + D\omega \cos \beta, \\
 C^2 &= v_x^2 + v_y^2 + 2D[pv_x \sin \beta + \omega v_y \cos \beta] \\
 &\quad + D^2[p^2 \sin^2 \beta + \omega^2 \cos^2 \beta].
 \end{aligned} \quad (20)$$

In what follows we retain only the terms of the lowest degree in the amplitudes A and B of the forces, on the assumption that the velocities acquired along the free paths are small compared with U.

The square of the orbital velocity is, at time t ,

$$\begin{aligned}
 \left(\frac{ds}{dt}\right)^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \\
 &= C^2 \sin^2(\omega t + \alpha) - 2CDp \sin(\omega t + \alpha) \sin(pt + \beta) \\
 &\quad + C^2 \cos^2(\omega t + \alpha) - 2CD\omega \cos(\omega t + \alpha) \cos(pt + \beta) \\
 &\quad + v_z^2 + 2v_z \frac{B}{p} [\sin(pt + \gamma) - \sin \gamma] + \dots, \\
 &\doteq U^2 + 2D[pv_x \sin \beta + \omega v_y \cos \beta] \\
 &\quad - 2CD[p \sin(\omega t + \alpha) \sin(pt + \beta) \\
 &\quad + \omega \cos(\omega t + \alpha) \cos(pt + \beta)] \\
 &\quad + 2v_z \frac{B}{p} [\sin(pt + \gamma) - \sin \gamma],
 \end{aligned}$$

whence

$$\begin{aligned}
 \left(\frac{ds}{dt}\right) &\doteq U + \frac{D}{U} [pv_x \sin \beta + \omega v_y \cos \beta] \\
 &\quad - \frac{CD}{2U} [(\omega + p) \cos[(\omega - p)t + \alpha - \beta] \\
 &\quad + (\omega - p) \cos[(\omega + p)t + \alpha + \beta]] \\
 &\quad + \frac{v_z B}{p U} [\sin pt \cos \gamma - \sin \gamma (1 - \cos pt)].
 \end{aligned}$$

Consequently, the distance travelled in the orbit in time t is

$$\begin{aligned}
 s = & Ut + \frac{D}{U} [pv_x \sin \beta + \omega v_y \cos \beta] t \\
 & + \frac{CD}{2U} \left\{ \left(\frac{\omega+p}{\omega-p} \right) [\sin(\alpha-\beta) - \sin((\omega-p)t + \alpha - \beta)] \right. \\
 & \quad \left. + \left(\frac{\omega-p}{\omega+p} \right) [\sin(\alpha+\beta) - \sin((\omega+p)t + \alpha + \beta)] \right\} \\
 & - \frac{v_z B}{p^2 U} [\cos \gamma \cos pt + (pt - \sin pt) \sin \gamma].
 \end{aligned}$$

The time t' required to traverse the distance s is (approximately)

$$t' \doteq \frac{s}{U} - A = \tau - A,$$

where $\tau = \frac{s}{U}$,

and

$$\begin{aligned}
 A = & \frac{CD}{2U^2} \left\{ \left(\frac{\omega+p}{\omega-p} \right) [\sin(\alpha-\beta) \{1 - \cos(\omega-p)\tau\} \right. \\
 & \quad \left. - \cos(\alpha-\beta) \sin(\omega-p)\tau] \right. \\
 & \quad \left. + \left(\frac{\omega-p}{\omega+p} \right) [\sin(\alpha+\beta) \{1 - \cos(\omega+p)\tau\} \right. \\
 & \quad \left. - \cos(\alpha+\beta) \sin(\omega+p)\tau] \right\} \\
 & + \frac{D^2}{U} [pv_x \sin \beta + \omega v_y \cos \beta] \tau \\
 & - \frac{v_z B}{p^2 U} [\cos \gamma \cos p\tau + \sin \gamma \cdot (p\tau - \sin p\tau)]. \quad \dots \quad (21)
 \end{aligned}$$

The work done on an electron over a free path s by the component X is

$$w_v = \int_0^{t'} X e \dot{x} dt = \int_0^{\tau} X e \dot{x} dt - e(X \dot{x} A),$$

We first evaluate

$$\begin{aligned}
 & \int_0^{\tau} X e \dot{x} dt \\
 & = e A \int_0^{\tau} [C \sin(\omega t + \alpha) - D p \sin(pt + \beta)] \cos(pt + \beta) dt,
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{eAC}{2} \int_0^\tau [\sin \{(\omega+p)t + \alpha + \beta\} + \sin \{(\omega-p)t + \alpha - \beta\}] dt \\
 &\quad - \frac{eAC}{2} \int_0^\tau \sin 2(pt + \beta) pt, \\
 &= \frac{eAC}{2} \left[\frac{\cos(\alpha + \beta)(1 - \cos(\omega + p)\tau) + \sin(\alpha + \beta)\sin(\omega + p)\tau}{(\omega + p)} \right. \\
 &\quad \left. + \frac{\cos(\alpha - \beta)(1 - \cos(\omega - p)\tau) + \sin(\alpha - \beta)\sin(\omega - p)\tau}{(\omega - p)} \right] \\
 &\quad + \frac{eADp}{4} [\cos 2(pt + \beta) - \cos 2\beta].
 \end{aligned}$$

The collisions are distributed evenly over the period $p/2\pi$ of an oscillation, so that the mean work done over paths of length s may be found by averaging for β from 0 to 2π .

The second term vanishes in the mean, while the first gives on substituting for C from Appendix B, equations (1) and (2)

$$\begin{aligned}
 &\frac{eAD}{4} \left[\left(\frac{\omega - p}{\omega + p} \right) (1 - \cos(\omega + p)\tau) + \left(\frac{\omega + p}{\omega - p} \right) (1 - \cos(\omega - p)\tau) \right] \\
 &= \frac{1}{4} \rho A^2 \left[\frac{1 - \cos(\omega + p)\tau}{(\omega + p)^2} + \frac{1 - \cos(\omega - p)\tau}{(\omega - p)^2} \right]. \quad . \quad (22)
 \end{aligned}$$

The evaluation of the correction term $e(X\dot{x}\Delta)_\tau$ is greatly facilitated by use of the relations tabulated in Appendix B.

We have

$$\begin{aligned}
 (X\dot{x})_\tau &= AC \cos(pt + \beta) \sin(\omega\tau + \alpha) \\
 &\quad - ADp \cos(pt + \beta) \sin(pt + \beta).
 \end{aligned}$$

In the product $(X\dot{x}\Delta)_\tau$ we reject terms in A , D , B , or their products, of degree higher than the second, consequently we may discard the second term in $(X\dot{x})_\tau$ as every term in Δ involves an amplitude.

$(X\dot{x})_\tau$ now becomes

$$\begin{aligned}
 (X\dot{x})_\tau &= \frac{AC}{2} [\sin\{(\omega + p)\tau + \alpha + \beta\} + \sin\{(\omega - p)\tau + \alpha - \beta\}], \\
 &= \frac{AC}{2} [\cos(\alpha + \beta) \sin(\omega + p)\tau + \sin(\alpha + \beta) \cos(\omega + p)\tau \\
 &\quad + \cos(\alpha - \beta) \sin(\omega - p)\tau + \sin(\alpha - \beta) \cos(\omega - p)\tau].
 \end{aligned}$$

The product $(X\dot{x}\Delta)_\tau$ will consequently consist of a great number of terms involving quantities of the type $C^2 \sin(\alpha + \beta) \sin(\alpha - \beta)$. However, when the mean is taken first with respect to β and then over all initial velocities v_x , v_y , and v_z , the only terms which remain are those of the form $C^2 \cos^2(\alpha + \beta)$ and terms in $\cos^2 \beta$ and $\sin^2 \beta$, as may be seen by reference to Appendix B.

We find, therefore, for the mean value of $(X\dot{x}\Delta)_\tau$, averaged over all phase angles β , and velocities v_x , v_y , v_z ,

$$\begin{aligned}
 (X\dot{x}\Delta)_\tau &= \frac{AD}{12} \left\{ \left(\frac{\omega - p}{\omega + p} \right) [\cos(\omega + p)\tau(1 - \cos(\omega + p)\tau) \right. \\
 &\quad \left. - \sin^2(\omega + p)\tau] \right. \\
 &\quad \left. + \left(\frac{\omega + p}{\omega - p} \right) [\cos(\omega - p)\tau(1 - \cos(\omega - p)\tau) \right. \\
 &\quad \left. - \sin^2(\omega - p)\tau] \right\} \\
 &\quad + \frac{AD\tau}{12} [(\omega - p) \sin(\omega + p)\tau + (\omega + p) \sin(\omega - p)\tau], \\
 &= \frac{AD}{12} \left\{ \left(\frac{\omega - p}{\omega + p} \right) (\cos(\omega + p)\tau - 1) \right. \\
 &\quad \left. + \left(\frac{\omega + p}{\omega - p} \right) (\cos(\omega - p)\tau - 1) \right. \\
 &\quad \left. + \tau[(\omega - p) \sin(\omega + p)\tau + (\omega + p) \sin(\omega - p)\tau] \right\}.
 \end{aligned}$$

It follows from (22), that

$$\begin{aligned}
 \bar{w}_{t'} &= e \int_0^\tau X\dot{x} dt - e(X\dot{x}\Delta)_\tau, \\
 &= \frac{eAD}{4} \left\{ \frac{4}{3} \left[\left(\frac{\omega - p}{\omega + p} \right) (1 - \cos(\omega + p)\tau) \right. \right. \\
 &\quad \left. + \left(\frac{\omega + p}{\omega - p} \right) (1 - \cos(\omega - p)\tau) \right] \\
 &\quad \left. - \frac{\tau}{3} [(\omega - p) \sin(\omega + p)\tau + (\omega + p) \sin(\omega - p)\tau] \right\}.
 \end{aligned}$$

The average is next taken for all free paths s , giving

$$\bar{w}_{t'} = \int_0^\infty \bar{w}_{t'} e^{-s/l} \frac{ds}{l} = \int_0^\infty w_{t'} e^{-\tau/T} \frac{d\tau}{T},$$

where

$$T = \frac{l}{U} \quad \text{and} \quad \tau = \frac{s}{U}.$$

We find for the mean energy expended per free path

$$\begin{aligned} \bar{w}_t = & \frac{eAD}{4} \left\{ \frac{4}{3} \left[\left(\frac{\omega+p}{\omega-p} \right) \left(\frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right. \right. \\ & \quad \left. \left. + \left(\frac{\omega-p}{\omega+p} \right) \left(\frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) \right] \right. \\ & \quad \left. - \frac{2}{3} \left[\frac{(\omega^2-p^2) T^2}{1+(\omega+p)^2 T^2} \left(1 - \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) \right. \right. \\ & \quad \left. \left. + \frac{(\omega^2-p^2) T^2}{1+(\omega-p)^2 T^2} \left(1 - \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right] \right\}, \\ = & \frac{2e^2}{3m} \left(\frac{A^2}{2} \right) \cdot \frac{T^2}{2} \left\{ \frac{1}{1+(\omega-p)^2 T^2} \left(1 + \frac{(\omega-p)^2 T^2}{1+(\omega-p)^2 T^2} \right) \right. \\ & \quad \left. + \frac{1}{1+(\omega+p)^2 T^2} \left(1 + \frac{(\omega+p)^2 T^2}{1+(\omega+p)^2 T^2} \right) \right\}. \end{aligned}$$

The mean rate of expenditure of energy by the field is, per electron,

$$\dot{w} = \frac{\bar{w}_t}{T},$$

and in the steady state this energy is dissipated in collisions between electrons and molecules of the gas.

The value of \bar{w} calculated in this way when averaged over all values of τ , is clearly identical with that given in equation (16).

The work done by the component Z is found similarly to be, per second per electron,

$$\bar{w}_z = \frac{2}{3} e \rho \left(\frac{B^2}{2} \right) \cdot \frac{1}{2} \left[\frac{T}{1+p^2 T^2} \left(1 + \frac{p^2 T^2}{1+p^2 T^2} \right) \right],$$

and is independent of H .

The actual numerical coefficient which appears in an expression for a velocity of drift W depends on the distribution of the quantities T about their mean value and on the particular average chosen for retention in the final equation.

For example^{(2), (1)}, the formula for the velocity W in a steady field X is

$$W = \frac{2}{3} \rho X \bar{T} = \frac{2}{3} \frac{Xe}{m} \left(\frac{\bar{l}}{\bar{U}} \right).$$

In the steady state the distribution function for the velocities U is such that

$$\frac{2}{3} \left(\frac{\bar{l}}{\bar{U}} \right) = \left(\frac{\bar{l}U}{U^2} \right).$$

Consequently, when l is independent of \bar{U} , W becomes

$$W = \frac{Xe}{m} l \left(\frac{\bar{U}}{U^2} \right).$$

Let the distribution function be supposed to be that of Maxwell, which is a possible function for the steady state of motion, then $\bar{U} = 92\sqrt{U^2}$ and W becomes

$$W = 92 \frac{Xe}{m} \cdot \frac{l}{\sqrt{U^2}}.$$

This is a useful formula, because the mean square velocity \bar{U}^2 is known in many cases from experiment⁽⁵⁾.

The formulæ for the velocities W in a magnetic field are more complicated, but the simplified forms which they assume when $(\omega \pm p)T \gg 1$ are of interest for the theories of the propagation of radio waves in the ionosphere.

In this case formulæ (14), (15), and (16) reduce to

$$\left. \begin{aligned} W_x &\doteq \frac{\rho A}{2} \left[\frac{1}{(\omega+p)} - \frac{1}{(\omega-p)} \right] \sin pt, \\ W_y &\doteq -\frac{\rho A}{2} \left[\frac{1}{(\omega+p)} + \frac{1}{(\omega-p)} \right] \cos pt. \end{aligned} \right\} . \quad (23)$$

$$\bar{w} \doteq \frac{4e^2}{3m} \left(\frac{A^2}{2} \right) \left(\frac{1}{2T} \right) \left[\frac{1}{(\omega+p)^2} + \frac{1}{(\omega-p)^2} \right] . \quad (24)$$

per electron.

We write

$$\left(\frac{\bar{l}}{\bar{T}} \right) = \left(\frac{\bar{U}}{l} \right).$$

Suppose, for example, that the mean free path l is independent of U , and that the velocities U are dis-

tributed according to Maxwell's law. We have, then,
 $\bar{U} = .92(\bar{U}^2)^{1/2}$

$$w \doteq 1.22 \frac{e^2}{ml} \left(\frac{A^2}{2} \right) \frac{\sqrt{\bar{U}^2}}{2} \left[\frac{1}{(\omega+p)^2} + \frac{1}{(\omega-p)^2} \right].$$

The actual distribution function in general is not that of Maxwell and the coefficient 1.22 would require slight modification.

The new general formula for \bar{w} leads to values somewhat greater than those given by formulæ⁽⁶⁾ used hitherto, when $(w \pm p) \gg 1$.

Appendix A.

Frequent use is made of the following definite integrals:—

$$\int_0^\infty e^{-\tau/T} \cos \omega \tau \, d\tau = \frac{T}{1+\omega^2 T^2},$$

$$\int_0^\infty e^{-\tau/T} (1 - \cos \omega \tau) \, d\tau = \frac{\omega^2 T^3}{1+\omega^2 T^2},$$

$$\int_0^\infty e^{-\tau/T} \sin \omega \tau \, d\tau = \frac{\omega T^2}{1+\omega^2 T^2},$$

$$\int_0^\infty e^{-\tau/T} \tau \cos \omega \tau \, d\tau = \left(\frac{T^2}{1+\omega^2 T^2} \right) \left(1 - \frac{2\omega^2 T^2}{1+\omega^2 T^2} \right),$$

$$\int_0^\infty e^{-\tau/T} \tau \sin \omega \tau \, d\tau = \frac{2\omega T^3}{1+\omega^2 T^2}.$$

Appendix B.

We have from (20),

$$C \sin \alpha = v_x + D p \sin \beta,$$

$$C \cos \alpha = v_y + D \omega \cos \beta,$$

whence

$$\left. \begin{aligned} C \sin(\alpha + \beta) &= v_x \cos \beta + v_y \sin \beta + \frac{D(\omega + p)}{2} \sin 2\beta, \\ C \sin(\alpha - \beta) &= v_x \cos \beta - v_y \sin \beta - \frac{D(\omega - p)}{2} \sin 2\beta, \\ C \cos(\alpha + \beta) &= -v_x \sin \beta + v_y \cos \beta + D(\omega \cos^2 \beta - p \sin^2 \beta), \\ C \cos(\alpha - \beta) &= v_x \sin \beta + v_y \cos \beta + D(\omega \cos^2 \beta + p \sin^2 \beta). \end{aligned} \right\} \quad \dots \quad (1)$$

The means of the following quantities with respect to β , on rejecting terms in D^2 , are,

$$\left. \begin{aligned} \text{mean: } C^2 \sin(\alpha + \beta) \sin(\alpha - \beta) &= \frac{v_x^2 - v_y^2}{2}, \\ C^2 \sin(\alpha + \beta) \cos(\alpha + \beta) &= 0, \\ C^2 \sin(\alpha + \beta) \cos(\alpha - \beta) &= v_x v_y, \\ C^2 \sin(\alpha - \beta) \cos(\alpha + \beta) &= v_x v_y, \\ C^2 \sin(\alpha - \beta) \cos(\alpha - \beta) &= 0, \\ C^2 \cos(\alpha + \beta) \cos(\alpha - \beta) &= -\frac{v_x^2 - v_y^2}{2}, \\ C^2 \cos^2(\alpha + \beta) \\ C^2 \sin^2(\alpha + \beta) \\ C^2 \cos^2(\alpha - \beta) \\ C^2 \sin^2(\alpha - \beta) \end{aligned} \right\} . \quad (2)$$

In averaging further with respect to v_x and v_y we write

$$\bar{v_x} v_y = \bar{v_x} \bar{v}_y = 0; \quad \bar{v_x^2} = \bar{v_y^2} = \frac{U^2}{3}.$$

Consequently, the only terms which remain are the last four which become each equal in the mean to $U^2/3$.

Similarly,

$$\left. \begin{aligned} \text{mean: } Cv_x \sin \beta \cos(\alpha + \beta) &= -\frac{v_x^2}{2}, \\ Cv_x \sin \beta \cos(\alpha - \beta) &= \frac{v_x^2}{2}, \\ Cv_x \sin \beta \sin(\alpha + \beta) &= 0, \\ Cv_x \sin \beta \sin(\alpha - \beta) &= 0, \\ Cv_y \cos \beta \cos(\alpha + \beta) &= \frac{v_y^2}{2}, \\ Cv_y \cos \beta \cos(\alpha - \beta) &= \frac{v_y^2}{2}, \\ Cv_y \cos \beta \sin(\alpha + \beta) &= 0, \\ Cv_y \cos \beta \sin(\alpha - \beta) &= 0. \end{aligned} \right\} \quad (3)$$

Summary.

In many theoretical treatments of the motions of electrons in gases in electric fields by the method of free paths, no account is taken of the effect of the electric field on the time spent between collisions of electrons with gas molecules.

In a previous paper an investigation was given of the motion of electrons in a gas under a steady electric field accompanied by a magnetic field, in which allowance was made for the effect of the field on the "free time" between collisions.

In the present paper the same methods are used to investigate the motion of electrons in gases, under the action of simple harmonic electric fields either with or without an accompanying constant magnetic field.

A general formula is obtained which contains several previous formulae as special instances.

The investigation is of interest for the study of the ionosphere by means of radio waves.

Note added in proof.—It may be seen as follows that this assumption leads to the correct result whatever the dependence of the field $Z(t)$ on the time.

With no field the fraction of electrons which have travelled distances between r and $r+dr$, and are moving in directions between $(\theta-\alpha)$ and $(\theta-\alpha+d\theta)$ is

$$\frac{1}{2}e^{-r/l} \frac{dr}{l} \sin(\theta-\alpha) d\theta.$$

In a field $Z(t)$ these electrons are replaced by the class previously travelling between θ and $(\theta+d\theta')$, where

$$d\theta' = \left(1 + \frac{d\alpha}{d\theta}\right) d\theta, \quad \alpha = \frac{\Delta v_z}{U} \sin \theta \text{ being the change in direction}$$

due to the field, from θ to $\theta-\alpha$, and Δv_z the velocity added parallel to OZ during the time spent in travelling the distance r .

Consequently, the number of electrons transported in time dt across dS within the angular range $d\theta$ about $(\theta-\alpha)$ becomes

$$\begin{aligned} & n \frac{dS}{2} e^{-r/l} \frac{dr}{l} U \cos(\theta-\alpha) \sin(\theta-\alpha) d\theta \left(1 + \frac{\Delta v_z}{U} \cos \theta\right) dt \\ & = n \frac{dS}{2} e^{-r/l} \frac{dr}{l} [U \cos \theta \sin \theta + \Delta v_z \sin^3 \theta] d\theta dt, \end{aligned}$$

in agreement with the result in the text.

As explained in the paper, when Z is a function of the time the interval dt is to be replaced (in the general case) by

$$dt' = dt \left(1 - \frac{\cos \theta}{U} \frac{d}{dt'} \int_{t'}^{t'+r/U} \Delta v_z dT \right)$$

in the integrand.

I am indebted to Mr. E. H. Saayman for pointing out that in the final equation (2a) in the first paper⁽¹⁾ the term in w , which was discarded as small, actually vanishes. My best thanks are also due to him for useful criticism.

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University College, Leicester.
September 17th, 1936.

XLIII. The Frequency of Transverse Vibration of a Loaded Fixed-Free Bar.—II. The Effect of the Rotatory Inertia of the Load. By R. M. DAVIES, M.Sc., F.Inst.P., Lecturer in Physics, University College of Wales, Aberystwyth *.

Notation.

ρ =Density of the bar.

E =Young's Modulus of the material of the bar.

l =Length of the bar.

k =Radius of gyration of the cross-section of the bar about an axis through the centre of gravity, perpendicular to the plane of motion.

A =Area of cross-section of the bar.

$M' = \rho A l$ =Mass of the bar.

M =Mass of the load fixed to the free end of the bar.

$c = M/M'$ =Ratio of the mass of the load to the mass of the bar.

* Communicated by Vice-Principal Gwilym Owen, M.A., D.Sc.

κ =Radius of gyration of the load about an axis through the point of attachment of the load to the bar and perpendicular to the plane of motion.

I =Moment of inertia of the load about this axis.

$$d = \frac{I}{M'l^2} = \frac{c\kappa^2}{l^2}.$$

n =Natural frequency of vibration of the loaded bar when vibrating in the fundamental mode.

$\omega = 2\pi n$ =Natural pulsatance of the loaded bar for the fundamental mode of vibration.

m =Frequency parameter, defined by the equation

$$m^4 = \frac{\rho\omega^2}{Ek^2}.$$

$$z=ml; \alpha=z^4.$$

THE problem discussed in the present paper arises from a dynamical method developed in this Laboratory⁽¹⁾ for the determination of Young's Modulus; in this method, the fundamental frequency, n , of transverse vibration of a loaded fixed-free bar is determined by experiment and from a knowledge of n and of some of the constants of the bar and the load, the Young's Modulus of the material of the bar can be calculated. To a first approximation, the load can be considered as a point mass, *i. e.*, a mass devoid of rotatory inertia; in this case, the equation connecting n and the constants of the bar is a transcendental equation⁽²⁾, and, using a series method, an explicit solution of this equation has been given in a previous paper⁽³⁾.

In general, however, the load will possess rotatory inertia, and the object of the present paper is to investigate how this will modify the solutions already given; it is also necessary to determine to what extent the moment of inertia of the load affects the calculation of E from experimental results.

The problem has been investigated by Rayleigh⁽⁴⁾ using the method of approximation which he devised for dealing with problems of this type. The case which he considers is that of a fixed-free bar, loaded at the free end; he assumes that the energy of motion of the bar itself is negligible in comparison with the energy of motion of the load, *i. e.*, that the bar can be considered as being devoid

of mass. In the present notation, the result for the fundamental mode may be written

$$\omega^2 = \frac{2EAk^2}{Mlk^2} \left\{ 1 + \frac{3\kappa^2}{l^2} - \sqrt{1 + \frac{3\kappa^2}{l^2} + \frac{9\kappa^4}{l^4}} \right\}, \quad (1)$$

or, in terms of z or α , since $\kappa^2 c = l^2 d$, and $z^4 = \alpha = m^4 l^4$

$$= \frac{\rho \omega^2 l^4}{E k^2},$$

$$\alpha_r = z_r^4 = \frac{2}{d} \left\{ 1 + \frac{3d}{c} - \sqrt{1 + \frac{3d}{c} + \frac{9d^2}{c^2}} \right\}, \quad . \quad (2)$$

where α_r and z_r denote the values of α and z given by this equation.

If d/c be small in comparison with unity, *i.e.*, if the rotatory inertia of the load be small in comparison with its mass, or if the radius of gyration of the load be small in comparison with the length of the bar, these equations can be simplified. Expanding the square root and neglecting cubes and higher powers of d/c , the equations become

$$\omega^2 = \frac{3EAk^2}{Ml^3} \left(1 - \frac{9d}{4c} \right); \quad \alpha_r = z_r^4 = \frac{3}{c} \left(1 - \frac{9d}{4c} \right). \quad (2a)$$

If, on the other hand, d/c be large in comparison with unity, then the factor of $2/d$ in equation (2) becomes

approximately $\left(1 + \frac{3d}{c} \right)^{\frac{1}{2}} \left\{ \left(1 + \frac{3d}{c} \right)^{\frac{1}{2}} - \left(\frac{3d}{c} \right)^{\frac{1}{2}} \right\}$, *i.e.*, $\frac{1}{2}$,

and thus

$$\omega^2 = \frac{EAk^2}{Mlk^2}; \quad \alpha_r = z_r^4 = \frac{1}{d}. \quad . \quad . \quad . \quad (2b)$$

Rayleigh's result can be extended to include the effect of the mass of the bar by using Dunkerley's Rule⁽⁵⁾. If ω_1 be the natural pulsatance of the unloaded bar (considered as possessing mass), ω_2 the natural pulsatance of the loaded bar (bar considered as being massless), and ω_d , the natural pulsatance of the loaded bar (bar considered as possessing mass), then by Dunkerley's Rule,

$$\frac{1}{\omega_d^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}.$$

Now, $\omega_1^2 = 12.364 \frac{Ek^2}{\rho l^4}$ and ω_2 is given by equation (1), hence

$$\frac{1}{\omega_d^2} = \frac{Mlk^2}{2EAk^2} \left\{ 1 + \frac{3d}{c} - \sqrt{1 + \frac{3d}{c} + \frac{9d^2}{c^2}} \right\} + \frac{\rho l^4}{12.364 Ek^2} \quad \dots \quad (3)$$

If z_d and α_d are the values of z and α corresponding to ω_d , we obtain

$$\frac{1}{\alpha_d} = \frac{1}{z_d^4} = \frac{1}{12.364} + \frac{d}{2} \cdot \frac{1}{\left\{ 1 + \frac{3d}{c} - \sqrt{1 + \frac{3d}{c} + \frac{9d^2}{c^2}} \right\}} \quad \dots \quad (4)$$

If d/c be small in comparison with unity, this equation becomes

$$\frac{1}{\alpha_d} = \frac{1}{z_d^4} = \frac{1}{12.364} + \frac{c}{3 \left(1 - \frac{9d}{4c} \right)} \quad \dots \quad (4a)$$

and, if d/c be large in comparison with unity, we have

$$\frac{1}{\alpha_d} = \frac{1}{z_d^4} = \frac{1}{12.364} + d \quad \dots \quad (4b)$$

Another approximate value for the natural frequency of a loaded fixed-free bar, taking into account both the mass of the bar and the rotatory inertia of the load, has been given by Morrow⁽⁶⁾. In the present notation, z_m , the value of z given by Morrow may be written

$$z_m^4 = \frac{1}{0.08194 + \frac{1}{3}c + \frac{2}{3}d} \quad \dots \quad (5)$$

It should be noted that this result is a first approximation and that it is possible to proceed to a second approximation which will yield a more accurate result.

Another method of deriving an expression for z in terms of c and d is to form the differential equation of motion of the bar and to find a solution which agrees with the boundary conditions at the free and fixed ends. The solution for the case where the load is devoid of rotatory inertia is given by Prescott⁽²⁾; the solution for the case

where the rotatory inertia of the load is not negligible does not appear to have been given, except for the case of a free-free bar⁽⁷⁾. The required solution may be derived as follows:—

Let the axis of the bar (assumed straight when in equilibrium) be taken as the axis of x , and let the direction of the displacement of the bar when vibrating be taken as the axis of y . If the rotatory inertia of the bar be neglected, the differential equation of motion is

$$\frac{\partial^2 y}{\partial t^2} + \frac{Ek^2}{\rho} \frac{\partial^4 y}{\partial x^4} = 0, \quad \dots \quad (6)$$

t denoting time.

For the case of a simple harmonic vibration of pulsatance ω , y will be of the form $y = ue^{i\omega t}$ where $i = \sqrt{-1}$ and where u (the normal or characteristic function) is a function of x only. This leads to the following differential equation for u :—

$$\frac{\partial^4 u}{\partial x^4} - m^4 u = 0, \quad \text{where } m^4 = \frac{\rho\omega^2}{Ek^2}. \quad \dots \quad (7)$$

The solution of this equation is

$$u = P \cosh mx + P' \sinh mx + Q \cos mx + Q' \sin mx, \quad \dots \quad (8)$$

where P, P', Q, Q' are integration constants determined by the boundary conditions.

At the fixed end of the bar, the displacement and the slope of the displacement curve must be zero; hence, if the fixed end be taken as the origin, we obtain $x=0$ and

$$u = \frac{\partial u}{\partial x} = 0 \text{ for the boundary conditions at the fixed end.}$$

Inserting these conditions in equation (8), we obtain

$$P + Q = P' + Q' = 0,$$

and equation (8) may be written

$$u = P(\cosh mx - \cos mx) + P'(\sinh mx - \sin mx). \quad \dots \quad (9)$$

Let the length of the bar be l . The conditions at the free end ($x=l$) are as follows, using the sign convention given in Prescott⁽²⁾, Chap. IX. :—

(i.) The product of the mass of the load into its linear

acceleration must be equal to the reversed shearing force. If F be the shearing force, then

$$-F = EAk^2 \frac{\partial^3 y}{\partial x^3} = M \frac{\partial^2 y}{\partial t^2}.$$

In terms of u , this equation becomes

$$-M\omega^2 u = EAk^2 \frac{\partial^3 u}{\partial x^3},$$

and finally

$$\frac{\partial^3 u}{\partial x^3} = -clm^4 u. \quad \dots \quad (10)$$

(ii.) The product of the moment of inertia I of the load into its angular acceleration is equal to the reversed bending moment. If M_b be the bending moment

$$M_b = +EAk^2 \frac{\partial^2 y}{\partial x^2} = -I \frac{\partial^2}{\partial t^2} \left(\frac{\partial y}{\partial x} \right),$$

which gives

$$\frac{\partial^2 u}{\partial x^2} = clm^2 m^4 \frac{\partial y}{\partial x} = m^4 l^3 d \frac{\partial u}{\partial x}. \quad \dots \quad (11)$$

If the values of $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^3 u}{\partial x^3}$ are found from equation (9), substitution in equation (10) gives

$$P[(\sinh ml - \sin ml) + cml (\cosh ml - \cos ml)] + P'[(\cosh ml + \cos ml) + cml (\sinh ml - \sin ml)] = 0,$$

whilst equation (11) gives

$$P[(\cosh ml + \cos ml) - dm^3 l^3 (\sinh ml + \sin ml)] + P'[(\sinh ml + \sin ml) - dm^3 l^3 (\cosh ml - \cos ml)] = 0.$$

Eliminating P and P' from these equations, we obtain the following equation for $z=ml$:

$$1 + \cosh z \cos z = \{cz(\cosh z \sin z - \sinh z \cos z) + dz^3(\cosh z \sin z + \sinh z \cos z) + cdz^4(\cosh z \cos z - 1)\}. \quad \dots \quad (12)$$

This is the required equation giving z in terms of c and d . It may be noted, that for an unloaded, fixed-free bar ($c=d=0$), it reduces to the well-known form

$$1 + \cosh z \cos z = 0,$$

whilst, if $d=0$, *i. e.*, neglecting the rotatory inertia of the load, it reduces to the form given by Prescott⁽²⁾ namely,

$$1 + \cosh z \cos z = cz (\cosh z \sin z - \sinh z \cos z).$$

If c is small in comparison with d , *i. e.*, if the mass of the load is small in comparison with its moment of inertia, equation (12) reduces to

$$1 + \cosh z \cos z = dz^3 (\cosh z \sin z + \sinh z \cos z).$$

For the purpose of calculating Young's Modulus from experimental results, it is necessary to obtain an explicit relationship giving z in terms of c and d . To do this the series method given in the previous paper⁽³⁾ may be used. Expanding the various terms of equation (12) by means of series⁽⁸⁾ and rejecting terms involving z^{16} and higher powers of z , we obtain

$$\left. \begin{aligned} 1 + \cosh z \cos z &= 2 - \frac{2^2}{4!} z^4 + \frac{2^4}{8!} z^8 - \frac{2^6}{12!} z^{12} + \dots \\ &= 2 - \frac{z^4}{6} + \frac{z^8}{2520} - \frac{z^{12}}{7,484,400} + \dots, \\ cz (\cosh z \sin z - \sinh z \cos z) &= 2cz \left[\frac{2}{3!} z^3 - \frac{2^3}{7!} z^7 + \frac{2^5}{11!} z^{11} - \dots \right] \\ &= c \left[\frac{2}{3} z^4 - \frac{z^8}{315} + \frac{z^{12}}{623,700} - \dots \right], \\ dz^3 (\cosh z \sin z + \sinh z \cos z) &= 2dz^3 \left[z - \frac{2^2}{5!} z^5 + \frac{2^4 z^9}{9!} + \dots \right] \\ &= 2d \left[z^4 - \frac{z^8}{30} + \frac{z^{12}}{11,340} + \dots \right], \\ cdz^4 (\cosh z \cos z - 1) &= cdz^4 \left(- \frac{2^2}{4!} z^4 + \frac{2^4}{8!} z^8 - \dots \right) \\ &= -cdz^8 \left(\frac{1}{6} - \frac{z^4}{2520} + \dots \right). \end{aligned} \right\} (13)$$

These series converge for all values of z .

Using the expressions of equation (13) and putting $z^4 = \alpha$, equation (12) becomes

$$2 - \frac{\alpha}{6}(1 + 4c + 12d) + \frac{\alpha^2}{2520}(1 + 8c + 168d + 420cd) - \frac{\alpha^3}{7,484,400}(1 + 12c + 660d + 2970cd) = 0. \quad (14)$$

Using an argument similar to that used in the previous paper and considering a range of possible values of c and d , it can be shown that the error committed in neglecting the α^3 term in comparison with the α term will not exceed 1 in 1000. We may thus neglect the α^3 term and the equation then becomes a quadratic equation in α . The smaller root of this equation corresponds to the fundamental mode of vibration, whilst the larger root corresponds to the first overtone; since the value of z for the first overtone is large, the above equation does not give a good approximation for this mode of vibration. Confining ourselves to the fundamental mode of vibration and writing

$$A = 1 + 8c + 168d + 420cd; \quad B = 1 + 4c + 12d,$$

we obtain

$$\alpha = z^4 = \frac{210}{A} \left(B - \sqrt{B^2 - \frac{4A}{35}} \right) \dots \quad (15)$$

or, in terms of c and d ,

$$\alpha = z^4 = \frac{210}{1 + 8c + 168d + 420cd} \left[(1 + 4c + 12d) - \sqrt{\left\{ 16(c + 3d)^2 + \frac{31}{35}(1 + 8c) + \frac{24d}{5}(1 - 10c) \right\}} \right] \dots \quad (16)$$

In the limiting cases of $c = d = 0$ and $d = 0$, this equation reduces to the forms given previously. If c is negligibly small, it becomes

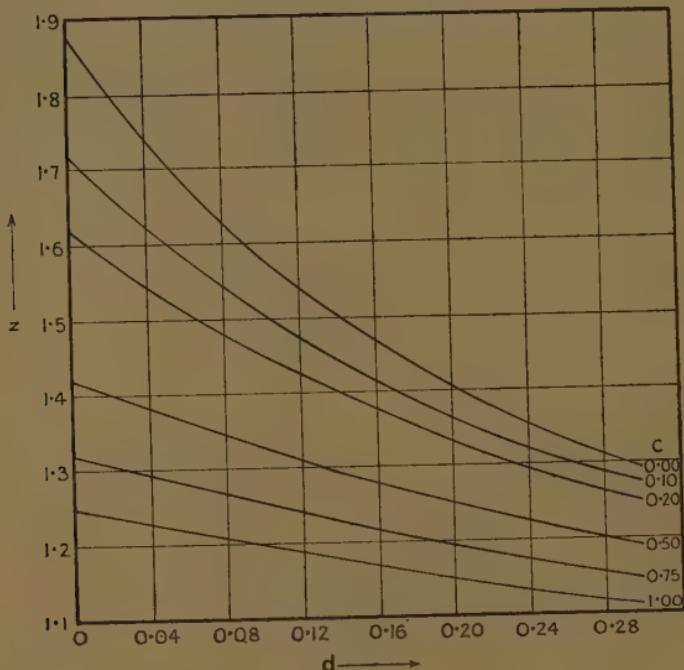
$$\alpha = z^4 = \frac{210}{1 + 168d} \left[(1 + 12d) - \sqrt{\left\{ \frac{31}{35} + \frac{24d}{5} + 144d^2 \right\}} \right].$$

The variation of z (as given by equations (15) and (16)) with d for values of $c = 0, 0.1, 0.2, 0.5, 0.75$, and 1.0 , is shown graphically in fig. 1, whilst the variation of z with c

for values of $d=0$, $0\cdot01$, $0\cdot1$, $0\cdot2$, and $0\cdot3$, is similarly shown in fig. 2. In these figures, c or d are plotted as abscissæ and z as ordinate.

The above values of z and the values of z given by the approximate formula of equation (4) can be compared by means of fig. 3; in this figure the full line curves give the value of z obtained from equations (15) and (16), whilst the dotted line curves show the values of z given

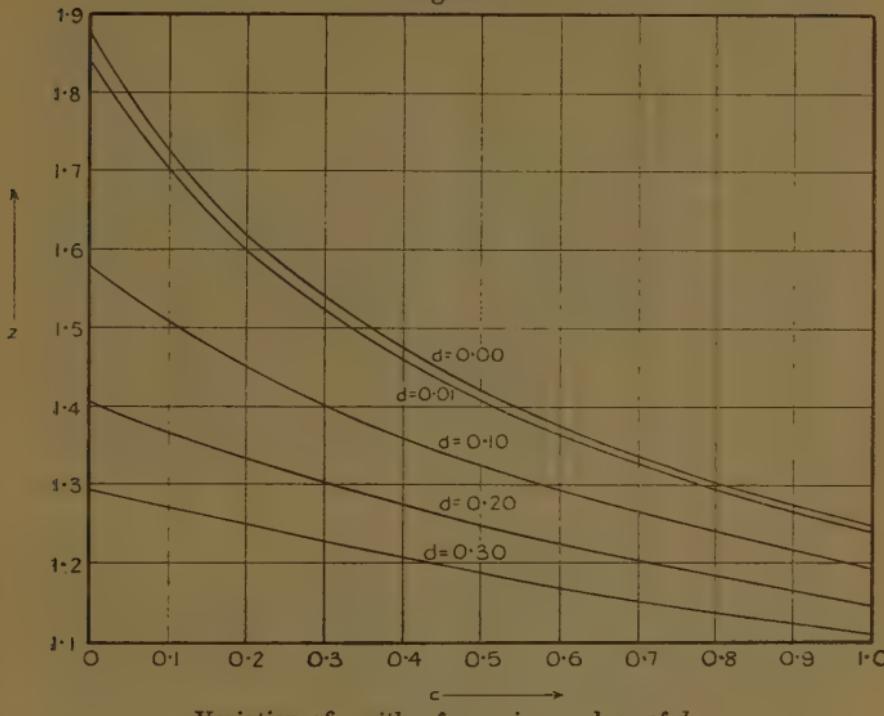
Fig. 1.

Variation of z with d for various values of c .

by equation (4). In this figure z is plotted as ordinate against d as abscissa for $c=0\cdot1$ and $c=0\cdot5$. It is clear from this figure that the difference between the two values of z decreases as c increases, and calculation shows that if $c=1\cdot0$, this difference is always less than 1 in 300 as d varies from 0 to $0\cdot3$.

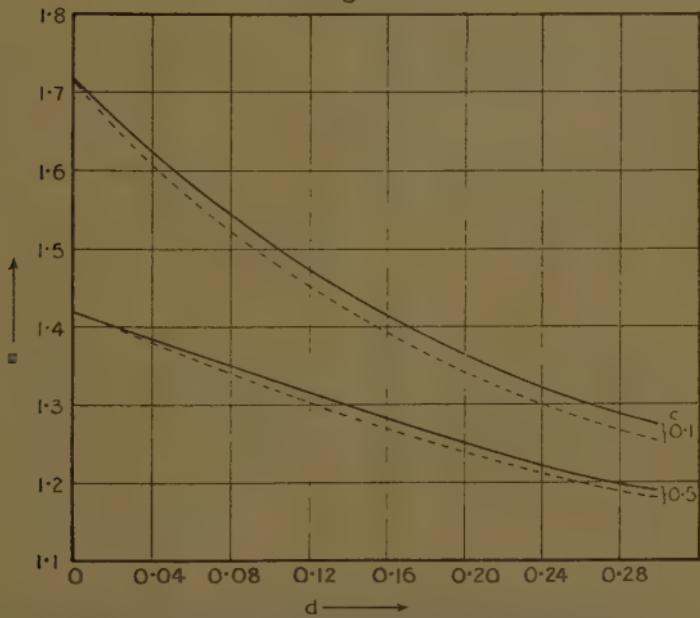
It remains to estimate the effect of the rotatory inertia of the load on the calculation of Young's Modulus in experiments such as those of Davies and James and Davies and Thomas⁽¹⁾. The magnitude of the effect

Fig. 2.



Variation of z with c for various values of d .

Fig. 3.



Comparison of the values of z given by accurate formula (full-line curve) and approximate formula (dotted-line curve).

may be illustrated by reference to some of the results given by Davies and Thomas ⁽¹⁾. We may first consider a result given for the first specimen of annealed Nickel, in which the length of the bar was 2.568 cm. and in which c was 0.03663; this corresponds to a comparatively small value of c , and if the rotatory inertia of the load be neglected, calculation gives $z^4=10.77$, and $z=1.812_0$. The load was a thin rectangular piece of stalloy, and if the length of this armature in a direction parallel to the direction of vibration be assumed to be 0.5 cm., d is found to be 1.157×10^{-4} . Using these values of c and d in equations (15) and (16), we obtain $z^4=10.77_1$ and $z=1.811_7$. Since Young's Modulus is inversely proportional to z^4 , the error due to neglect of rotatory inertia in this case is of the order of 0.08 per cent., which is less than the experimental error.

We may consider next a result given for unannealed Nickel in which the length was 2.321 cm. and in which c was 0.1153; this corresponds to a comparatively large value of c , and if the rotatory inertia of the load be neglected, z^4 is found to be 8.435_5 and $z=1.704_2$. If the dimensions of the load are assumed to be as before, $d=4.458 \times 10^{-4}$ and, from equations (15) and (16), $z^4=8.414_6$ and $z=1.703_2$; in this instance, neglect of the rotatory inertia of the load introduces an error of about 0.25 per cent. in z^4 , *i.e.*, in the Young's Modulus, and this exceeds the experimental error. If, however, the length of the stalloy armature in the direction of vibration is taken to be 0.3 cm., then $d=1.605 \times 10^{-4}$ and from equations (15) and (16), $z^4=8.427_9$ and $z=1.7038$; in this case the values of z differ by about 0.09 per cent., and a discrepancy of this magnitude is of the same order as the experimental error.

In the experimental results given in the papers quoted above, the length of the stalloy armature in the direction of vibration did not exceed 0.3 cm. in any experiment, and, in addition, the value of c was usually less than that used in the preceding paragraph; we may therefore conclude that, in these results, the effect of the rotatory inertia of the load may be neglected in calculating z .

In conclusion, the author desires to express his sincere thanks to Vice-Principal Gwilym Owen for his interest and encouragement and to Mr. Thomas Lewis, Head of

the Department of Applied Mathematics, for checking the results.

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October 1936.

XLIV. *On the Forces occurring in Dynamical Systems where the Obliquity of the Coordinate Axes varies with the Time, with an Application to the Theory of Electrical Machinery.* By W. H. INGRAM †.

SUMMARY.

A GENERALIZATION of the Hamel-Boltzmann-Lagrange formula is given which holds for dynamical systems in which the relative inclination of the coordinate axes varies with the time. A practical application to the theory of rotary electrical machinery is made and the equations of motion found for the 4-brush commutator motor for the case where one or both brush pairs are rotated in any manner.

JUST as a characteristic force, the so-called Coriolis force, appears when the reference frame of a dynamical system rotates, so a characteristic force, *the force of varying obliquity*, appears when the mutual obliquity of the coordinate axes of a dynamical system varies with the time.

When the directions of the coordinate axes vary independently, forces of both kinds occur and may be distinguished when that part of the change of direction due

† Communicated by the Author.

to rotation of the system of axes as a whole and that part due to varying obliquity mutual to members of the system are known.

The Equations of Motion.

Consider a system of rectilinear coordinates $x_1, x_2, x_3 \dots x_n$ of n dimensions where each coordinate axis has a direction which varies with the time when referred to a fixed or true system of coordinates $q_1, q_2, \dots q_n$. A point in the latter space represents n coordinates of a dynamical system where the equations of motion are given by the Lagrangean formula.

Considering the two coordinate systems as having a common origin, it is clear that the direction \dagger of the j th x -axis is given by $n-1$ angular coordinates $\alpha_{j1}, \alpha_{j2}, \dots \alpha_{j, j-1}, \alpha_{j, j+1}, \dots \alpha_{jn}$ suitably chosen on arcs on the hypersphere $\Sigma dg_i^2 = dr^2 = \text{const.}$

Accordingly, in the differential transformation \ddagger

$$dx_r = a_{r1} dq_1 + a_{r2} dq_2 + \dots + a_{rn} dq_n. \quad \dots \quad (1)$$

\dagger The set of $n(n-1)$ α 's, if taken as a measure of obliquity, is not invariant (it has, of course, an integer invariant) under transformations from one system of true coordinates to another. As Prof. Synge, who has been so kind as to read the MS of this paper, points out to me, the location of arcs on the hypersphere mentioned introduces an arbitrary element not intrinsic to the problem. I have not attempted to give an invariantive definition of obliquity and find that an invariantive definition of angle complicates the analysis without any compensating advantages. The ω 's have not been defined in terms of the α 's, as is possible, but have been left arbitrary for the reason that in the applications there is usually some practical consideration which determines the ω 's, and hence the not necessarily unique resolution (9).

\ddagger When the right-hand side of (1) is a not completely integrable Pfaffian, the space of the x 's is said (*vide* Vranceanu, "Les espaces non holonomes," Memorial des Sciences Mathématiques, fascicule lxxvi., Paris, 1936, p. 18) to form a "non-holonomic" sub-space associated with, or immersed in, the Riemannian space of the q 's defined by the usual kinematical line-element. In the case of any electrical machine, the coefficients in (1) are functions of a single dynamical coordinate θ (as well as the obliquity coordinates α_{ij}) which may be taken both as the $n+1$ st coordinate of the q -system and of the x -system; hence the metric $ds^2 = 2Tdt^2$ can be expressed completely in terms of the x 's and dx 's. The coefficients in the quadratic form giving the kinetic energy are components of a covariant tensor of the 2nd order for admissible q -systems of coordinates and for admissible x -systems respectively, but *only* respectively, and things that are tensors in one system are not tensors in both, in general. This fact makes any tensor formulation of the theory of electrical machinery, when sufficiently general to include both slip-ring and commutator machines, largely barren of tensors and correspondingly formal and without point.

each coefficient is a function of the $n(n-1)$ angular coordinates α_{ij} , where $i, j=1, 2, \dots, n$; $i \neq j$.

In the true coordinates the equation of motion of a forced, dissipative, dynamical system is given by the formula

$$\frac{\partial(U-S)}{\partial \dot{q}_k} = \frac{d}{dt} \left(\frac{\partial(T-V)}{\partial \dot{q}_k} \right) - \frac{\partial(T-V)}{\partial \dot{q}_k},$$

which applies to constrained systems when the proper "null-functions" [†] are added to the function U . We shall assume that the potential energy V is not a function of all the q 's and that (1) applies to coordinates not appearing [‡] in V . Moreover, we shall assume that T is not a function of any of the q 's of (1), but of a single coordinate $\theta = q_{n+1}$ which, in the application to follow, will be the angular coordinate of a machine rotor :

$$T = T(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, \dot{\theta}, \theta),$$

so that the equations of motion are

$$\left. \begin{aligned} \frac{\partial(U-S)}{\partial \dot{q}_k} &= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right), \quad k=1, 2, \dots, n, \\ \frac{\partial(U-S)}{\partial \dot{\theta}} &= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta}. \end{aligned} \right\} \quad \dots \quad (2)$$

Next consider the sum

$$\sum b_{kr} \frac{\partial(U-S)}{\partial \dot{q}_k} = \sum b_{kr} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) \quad \dots \quad (3)$$

where the b_{ij} 's are the coefficients in the inverse transformation

$$dq_k = b_{k1}dx_1 + b_{k2}dx_2 + \dots + b_{kn}dx_n, \quad \dots \quad (4)$$

obtained by solving the system (1) for dq_k . Equations (1) and (4) may be written

$$\left. \begin{aligned} \dot{x}_r &= \sum a_{ri} \dot{q}_i, \\ \dot{q}_k &= \sum b_{kj} \dot{x}_j. \end{aligned} \right\} \quad \dots \quad (5)$$

[†] Phil. Mag. xxi. p. 303 (1936).

[‡] This is no essential restriction, as coordinates appearing in V always can be separated from those whose time derivatives appear in T by enlarging the dynamical system artificially by plural designation of coordinates and including the forces of constraint which correspond to the equations of identity connecting the variables in the divided system.

where, from the properties of determinants

$$\sum a_{sk} b_{kr} = \begin{cases} 0, & r \neq s, \\ 1, & r = s. \end{cases} \quad \dots \quad (6)$$

Now, since

$$\frac{\partial T}{\partial \dot{x}_k} = \sum \frac{\partial T}{\partial \dot{x}_s} a_{sk} \quad (k, s = 1, 2, \dots, n) \quad \dots \quad (7)$$

the equations (3), making use of (6), become

$$\frac{\partial (U - S)}{\partial \dot{x}_r} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_r} \right) + \sum \sum b_{kr} \frac{\partial T}{\partial \dot{x}_s} \dot{a}_{sk} \quad \dots \quad (8a)$$

The last term in (8a) gives the force appearing when the directions of the coordinate axes are varying and includes the Coriolis force as a special case.

When this variation is resolvable into a part due to rotation of the x -frame as a whole about some $n-2$ sub-space of the x -space and into a part due to variation in the mutual obliquity of the members of the frame, a corresponding resolution of the forces is possible. Let the components of angular velocity of the x -frame about a given $n-2$ sub-space be $\omega_{12}, \omega_{13}, \dots, \omega_{n-1, n}$, where $\omega_{ij} = -\omega_{ji}$. Letting $\theta_{ij} = \int \omega_{ij} dt$ and $\alpha_{ij} - \theta_{ij} = \sigma_{ij}$, the last term in (8a) then may be written

$$\sum \sum b_{kr} \frac{\partial T}{\partial \dot{x}_s} \dot{a}_{sk} = \sum \sum \sum b_{kr} \left(\frac{\partial a_{sk}}{\partial \theta_{ij}} \omega_{ij} + \frac{\partial a_{sk}}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \right), \quad (9)$$

where the second term on the right-hand side expanded is the force of varying obliquity and where the first term is the Coriolis force \dagger .

It is understood that in (8a) T is expressed in terms of the \dot{x} 's and, when so expressed, we may write \ddot{x}

$$T = T^*(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, \dot{\theta}, \theta). \quad \dots \quad (10)$$

In the case of electrical machinery, the α_{ij} 's and hence the a_{ij} 's involve θ , the $n+1$ st coordinate. Hence

$$\frac{\partial T}{\partial \theta} = \frac{\partial T^*}{\partial \theta} + \sum \frac{\partial T^*}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \theta}, \quad \dots \quad (11)$$

\dagger Cf. Whittaker, 'Dynamics,' 3rd Ed., p. 41. The force ordinarily called after Coriolis is linear in the angular velocity of the coordinate system: it seems appropriate to retain the name in the present more general case.

\ddot{x} I have changed Whittaker's bar to an asterisk as more convenient to print.

since the \dot{x} 's also depend upon θ in this case. Corresponding, then, to the second equation of (2), one may write

$$\frac{\partial(U-S)}{\partial\dot{\theta}} = \frac{d}{dt}\left(\frac{\partial T^*}{\partial\dot{\theta}}\right) - \frac{dT^*}{d\theta}, \dots \quad (8b)$$

where the last term stands for the right-hand side of (11). The more conventional notation $\partial T^*/\partial\theta$ may be employed instead of $dT^*/d\theta$ only when the privileged character of the coordinate θ is borne in mind.

Application to Electrical Machinery.

A practical application of the previous theory arises when one pair of brushes of a 4-brush commutator motor is in motion with respect to the other pair. When the

Fig. 1.

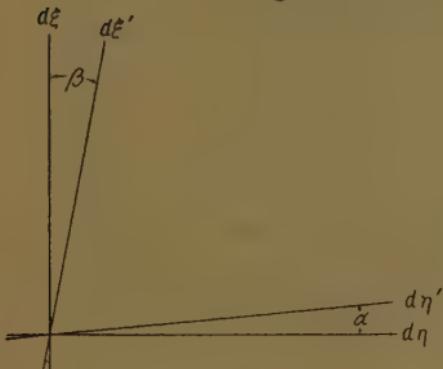
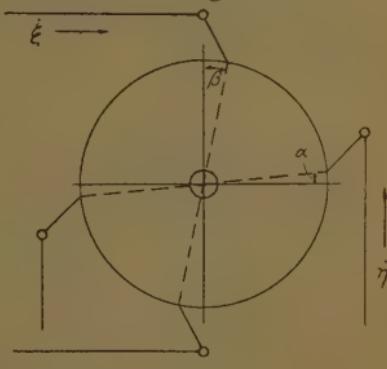


Fig. 2.



brush pairs are in the polar and inter-polar positions respectively, the true armature coordinates are given by the transformation

$$dq_i = \sqrt{\frac{2}{3}} \sin \theta_i d\xi + \sqrt{\frac{2}{3}} \cos \theta_i d\eta + \sqrt{\frac{1}{3}} d\zeta \quad (i=1, 2, 3), \quad (12)$$

where $d\xi$, $d\eta$, $d\zeta$ are the differentials of quasi-coordinates and may be taken as the axes of the ellipsoid previously discussed †.

A transformation to skew coordinates in the plane of the polar and inter-polar axes, with reference to fig. 1, is given by the equations

$$\begin{aligned} d\xi &= \cos \alpha d\xi' + \sin \beta d\eta', \\ d\eta &= \sin \alpha d\xi' + \cos \beta d\eta', \end{aligned} \quad \dots \quad (13)$$

† Phil. Mag. xxi. p. 301 (1936); the arrows shown indicate, of course, the relative motion of the two coordinate systems of which the q -system is stationary.

which may be substituted into (12) to give

$$dq_i = \sqrt{\frac{2}{3}} \sin(\theta_i + \alpha) d\xi' + \sqrt{\frac{2}{3}} \cos(\theta_i - \beta) d\eta' + \sqrt{\frac{1}{3}} d\zeta \quad (14)$$

connecting the physical coordinates, for the case where the brushes are inclined, with the true coordinates.

The kinetic energy, for the case of inclined brushes, is found to be †

$$\begin{aligned} T^* = & \frac{1}{2} (l^* + \frac{3}{2} l_\mu \cos 2\alpha) \dot{\xi}^2 + \frac{1}{2} (l^* - \frac{3}{2} l_\mu \cos 2\beta) \dot{\eta}^2 \\ & + \{l^* \sin(\alpha + \beta) - \frac{3}{2} l_\mu \sin(\alpha - \beta)\} \dot{\xi} \dot{\eta} + \frac{1}{2} (l^* - 3l_m) \dot{\zeta}^2 \\ & + M_0 \cos \alpha I \dot{\xi} + M_0 \sin \beta I \dot{\eta} + \frac{1}{2} L I^2 + \frac{1}{2} \Theta \dot{\theta}^2, \end{aligned} \quad \dots \quad (15)$$

where the primes have been dropped and where unessential terms (amortisseur currents, etc.) have been omitted.

Since ξ , η , ζ may be taken to correspond with x_1 , x_2 , x_3 of the first part of this paper, we have

$$\alpha_{12} = \alpha + \theta, \quad \alpha_{21} = \beta - \theta,$$

$$\alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = 0,$$

and the equations of motion are found to be

$$\left. \begin{aligned} \frac{\partial(U - S)}{\partial \dot{\xi}} &= \frac{d}{dt} \left(\frac{\partial T^*}{\partial \xi} \right) - (\dot{\theta} - \dot{\beta}) \frac{\partial T^*}{\partial \dot{\eta}} \cos(\alpha + \beta), \\ \frac{\partial(U - S)}{\partial \dot{\eta}} &= \frac{d}{dt} \left(\frac{\partial T^*}{\partial \eta} \right) + (\dot{\theta} + \dot{\alpha}) \frac{\partial T^*}{\partial \dot{\xi}} \cos(\alpha + \beta), \\ \frac{\partial(U - S)}{\partial \dot{\zeta}} &= \frac{d}{dt} \left(\frac{\partial T^*}{\partial \zeta} \right), \\ &\vdots & \vdots \\ \frac{\partial(U - S)}{\partial \dot{\theta}} &= \frac{d}{dt} \left(\frac{\partial T^*}{\partial \theta} \right) - \frac{d T^*}{d \theta}. \end{aligned} \right\} \quad (16)$$

These are the equations of voltage and torque for a 4-brush commutator machine where one pair of brushes is rotating with an angular velocity $\dot{\alpha}$ in a direction counter to that of the commutator segments and the other pair

† Allowing for a trivial change of scale and notation, this expression is the same as that given on p. 848 of the Phil. Mag. for April 1934 (where α has a special meaning) when $\alpha = -\beta$. An expression for the kinetic energy in terms of quasi-coordinates was first obtained early in 1933 and incorporated in a criticism appended to a paper by G. Kron, which had been accepted by the American Institute of Electrical Engineers for publication but which has never been published.

is rotating with an angular velocity $\dot{\beta}$ in the same direction as that of the commutator segments.

When $\dot{\alpha} = -\dot{\beta}$, the equations become those previously obtained †; the terms then in $\dot{\theta} - \dot{\beta}$, I have called the Coriolis e.m.f.'s ‡. In the present case, the Coriolis e.m.f. is $(\dot{\theta} + \frac{1}{2}(\dot{\alpha} - \dot{\beta})) \frac{\partial T^*}{\partial \eta} \cos(\alpha + \beta)$ and the force of varying obliquity is $-\frac{1}{2}(\dot{\alpha} + \dot{\beta}) \frac{\partial T^*}{\partial \eta} \cos(\alpha - \beta)$ for the first brush circuit, and a similar resolution of forces is possible for the second; obviously the mean angular velocity of the reference frame in this case is $\dot{\theta} + \frac{1}{2}(\dot{\alpha} - \dot{\beta})$.

When α and β are constants, the equations of motion of the Déri motor of fig. 2 are obtained.

For systems of machines connected in any manner, the system (16) still applies when one substitutes

$$\begin{array}{ll} T_1 + T_2 + T_3 + \dots & \text{for } T, \\ V_1 + V_2 + V_3 + \dots & \text{for } V, \\ S_1 + S_2 + S_3 + \dots & \text{for } S, \\ U_1 + U_2 + \dots + N_{1,2,\dots} & \text{for } U, \end{array}$$

where the subscripts refer to the various machines and where $N_{1,2,\dots}$ is a null-function of inter-constraint.

XLV. The Equations of Motion of Electrons in Gases.
By J. S. TOWNSEND, F.R.S., Wykeham Professor of Physics, Oxford §.

IN the investigations of the motion of ions and of electrons in gases under the action of an electric force, it is usually assumed that the equations of motion are of the same form as those in Maxwell's theory of inter-diffusion of gases.

In accordance with this theory, the mean velocity of charged particles of any mass in the directions of the

† Ingram, Phil. Mag. xvii. p. 848 (1934); xxi. p. 305 (1936).

‡ Bull. Am. Math. Soc. xl. p. 34 (1934); Proc. Nat. Academy, xx. p. 23 (1934) (tensor notation was used in these two notices, but it has since appeared that no tensors are involved in any significant way); Phil. Mag. *ibid.* p. 848.

§ Communicated by the Author.

coordinate axes x , y , and z are given by three equations of the form

$$Pw/K = -dP/dz + neZ, \dots \dots \dots \quad (1)$$

w being the mean velocity of the particles in the direction of the axis z , Z the electric force in that direction, n the number of particles per cubic centimetre of the gas, e the atomic charge, P the partial pressure of the particles, and K the coefficient of diffusion.

The equations of motion in this form require that there should be a simple relation between the coefficient of diffusion and the mean velocity W due to the electric force, which is $W \cdot K = 3Ze/mV^2$, where $mV^2/2$ is the mean energy of agitation of the particles.

This relation is satisfied by formulæ obtained by Langevin for the velocity W and the coefficient of diffusion K of ions in terms of the dimensions of the ions, taking into consideration the distribution of the velocities of agitation of the ions.

In a recent paper * on the distribution of the energies of electrons, I have shown that equation (1) is consistent with the expressions for the coefficient K and the velocity W of electrons in terms of the mean free path and the mean velocity of agitation of the electrons when the distribution of the energies of the electrons satisfies a certain condition. The Maxwellian distribution is of this form, but in general the velocities of agitation of the electrons are not distributed as widely as in Maxwell's distribution.

There is no simple expression for the distribution of the energies of the electrons, as it depends on the ratio of the mean energy of agitation of the electrons to that of the molecules of the gas, on the change in the mean free path with the energy of the electron, and on the mean loss of energy of an electron in a collision with a molecule.

The numerical factors that occur in the expressions for K and W in terms of the mean free path and the mean velocity of agitation depend on the distribution of the velocities, and in general the ratio W/K differs by a numerical factor from that required to satisfy equation (1).

It appears, therefore, that the mean velocities of electrons in a gas are not given accurately by equations of the

* Phil. Mag. xxii. p. 145 (July 1936).

form of equation (1), and it may be seen that these equations are obtained by an approximate calculation which gives rise to a noticeable error when the charged particles are of a mass which is small compared with that of a molecule of the gas.

The experiments on the lateral diffusion of a stream of electrons moving in a gas under the action of a uniform electric force show that the electrons attain a steady state of motion where the mean energy of agitation of the electrons remains constant as the stream moves in the direction of the force. For simplicity this investigation is limited to that type of motion where the mean energy of the electrons $mV^2/2$ is the same at all points of the gas.

The partial pressure P of the electrons is $mnV^2/3$, m being the mass of an electron, and when the energy $mV^2/2$ is constant, equation (1) may be simplified by substituting $(mV^2/3)dn/dz$ for dP/dz , and on dividing by $mV^2/3$ the equation for the mean velocity w becomes

$$nw/K = -dn/dz + 3neZ/mV^2. \dots \dots \quad (2)$$

An equation of motion of this form may also be obtained directly by adding the mean velocity $-Kdn/dz$ due to diffusion to the mean velocity W due to the electric force *. For this purpose it is necessary to determine K and W in terms of the mean free path l and the velocity V , taking into consideration the distribution of the velocities of agitation. In these calculations it may be assumed that after a collision with a molecule all directions of motion of an electron are equally probable, and that electrons may lose energy in small amounts in collisions with molecules which is restored by the displacements of the electrons in the direction of the electric force. Under these conditions the coefficient of diffusion K_1 of a group of electrons all moving with the same velocity of agitation U_1 is found to be $lU_1/3$, and the velocity \dagger W_1 to be $2Zel/3mU_1$.

When the velocities of agitation are distributed about the mean velocity \bar{U} , the mean value of K for all the electrons is obtained by substituting U for U_1 in the above expression for K_1 and the mean of the reciprocals of the velocities $(1/\bar{U})$ for $1/U_1$ in the expression for W_1 , assuming

* 'Electricity in Gases,' p. 93.

† Phil. Mag. xxii. p. 145 (July 1936).

for simplicity that the mean free path l is the same for all values of U . The mean velocity \bar{U} is less than the root-mean-square of the velocities V and the mean of the reciprocals $(1/\bar{U})$ is greater $1/V$, so that for a given value of the mean energy of the electrons, the wider the distribution of the energies the greater the value of the ratio W/K .

In general let the mean velocity of agitation be aV , and the mean of the reciprocals b/V , a and b being numerical factors depending on the distribution, so that the mean coefficient of diffusion is $laV/3$, the mean velocity W is $2Zelb/3mV$, and the ratio W/K is $2Zeb/amV^2$. The mean velocity w is the sum of the velocities $-Kdn/ndz$, due to diffusion, and W due to the electric force, and according to these calculations nw/K is given by the equation

$$nw/K = -dn/dz + 2neZb/amV^2, \dots \quad (3)$$

which agrees with equations (1) and (2) only when the numerical factor b is $3a/2$ (as in the Maxwellian distribution, where a is $.92$ and b is $.92 \times 3/2$).

Equation (3) is more general than equations (1) and (2), and it remains to be shown that equation (1) is based on a simplification which may introduce an appreciable error in the second term on the right of the equation.

That equation for w , which is in terms of the partial pressure of the electrons, is obtained by expressing the condition that the momentum of the electrons in an elementary space of volume $dx dy dz$ is constant in the steady motion obtained when a continuous stream of electrons moves through the gas under the action of a uniform force.

The normal to two sides of the elementary space which are of area $dx dy$ is in the direction of the axis z , and the rate at which momentum in that direction flows into the elementary space through the side nearest the plane $z=0$ due to the motion of agitation of the electrons is $P dx dy$, and the rate at which the momentum flows out of the space through the opposite side is $(P + dP/dz) dx dy dz$.

Thus the rate of increase of the momentum due to the diffusion through the two opposite sides is $-(dP/dz) dx dy dz$. The rate of increase in the momentum due to the action of the force Z is neZ , so that the total increase of momentum per second due to diffusion and to the force is

$$(-dP/dz + neZ) dx dy dz,$$

and in the steady motion of a continuous stream of electrons this increase of momentum is balanced by the loss of momentum in the collisions of electrons with molecules.

In order to find the loss of momentum in collisions it is necessary to distinguish between the momentum due to diffusion and the momentum acquired under the action of the electric force.

The mean velocity w of all the electrons normal to a plane due to diffusion is very small compared with the mean velocity of agitation \bar{U} , and nv is the difference between the numbers of electrons flowing per second in opposite directions through unit area of the plane, n being the total number of electrons in unit volume.

Let n_1 be the number in unit volume with velocities of agitation which are approximately equal to U_1 , and w_1 the mean velocity of the group n_1 in the direction z , n_2 , the number with velocities of agitation approximately equal to U_2 , and w_2 the mean velocity in the direction z of the group n_2 , etc. The velocities w_1, w_2 , etc., are proportional to the velocities U_1, U_2 , etc., so that the ratios $w_1/U_1, w_2/U_2$, etc., are equal to the ratio w/\bar{U} .

If l be the mean free path of an electron, the number of collisions per second between the electrons in the group n_1 and molecules of the gas is $n_1 U_1 / l$, and the loss of momentum of the electrons in these collisions is $mn_1 U_1 w_1 / l$, which may be written as $mn_1 U_1^2 \times w_1 / l \bar{U}_1$, or $mn_1 U_1^2 w / l \bar{U}$. The mean loss of momentum of the electrons in the group n_2 is $mn_2 U_2^2 w / l \bar{U}$, and the mean loss of momentum of all the electrons is $mnV^2 w / l \bar{U}$, where V^2 is the mean of the squares of the velocities of agitation. The mean loss of momentum of all the electrons is therefore Pw/K , where K is $l \bar{U} / 3$, so that the coefficient of diffusion obtained by this method is the same as that obtained by other methods *.

When the momentum in the direction z is due to the electric force, it does not follow from the above calculation that the loss of momentum in collisions is PW/K per second.

The mean velocities of any group of electrons in the directions of the axes are zero at the beginning of the free paths, so that if a number n' of electrons have the same velocity of agitation U' at the beginning of the free paths

* 'Electricity in Gases,' Section 81.

of length l' , the momentum at the end of the paths is $n_1 Z e l' / U'$, and this amount is lost in the collisions that terminate the free paths. Thus the amount lost in collisions is twice the mean momentum of the group n' during the time the electrons are moving in the free paths.

The mean loss of momentum per second of the group n' is $n' Z e$, and this mean loss per second of all the electrons is $n Z e$. This quantity is independent of the distribution of the energies and cannot in general be equal to PW/K , since both K and W involve numerical factors which depend on the distribution of the energies.

Equation (3), which is the general equation for the mean velocity w for any distribution of the energies of agitation of the electrons, may be written in the form

$$nw/K = -dn/dz + 3neZ/mV_1^2 \quad \dots \quad (4)$$

by substituting V_1^2 for $3aV^2/2b$, and if there be no forces in the directions of the axes x and y , the mean velocities u and v in those directions are given by the ordinary equations $nu/K = -dn/dx$ and $nv/K = -dn/dy$, and when these expressions are substituted for nu , nv , and nw in the equation of continuity, $d(nu)/dx + d(nv)/dy + d(nw)/dz = 0$, the differential equation for n becomes

$$\frac{d^2n}{dx^2} + \frac{d^2n}{dy^2} + \frac{d^2n}{dz^2} = \frac{3eZ}{mV_1^2} \cdot \frac{dn}{dz} \quad \dots \quad (5)$$

The energy $mV_1^2/2$ may be expressed by the potential E_1 which is $mV_1^2/2e$, and in this notation the term on the right of equation (5) is $1.5(Z/E_1)dn/dz$, Z being in volts per centimetre and E_1 in volts.

Thus the equation for n is of the same form for all distributions of the energies of the electrons, and the lateral divergence of a stream of electrons may be calculated by means of equation (5) in terms of the ratio Z/E_1 .

The energy E_1 which is thus obtained from the results of the experiments on the lateral divergence of a stream is equal to the mean energy of the electrons when the distribution of the energies is such that the coefficient b is equal to $3a/2$. For other distributions E_1 is $(3a/2b)mV^2/2$.

In general, when the mean energy of the electrons is greater than the energy corresponding to a potential of one volt, the coefficient b is less than $3a/2$ and the energy E_1 is somewhat greater than the mean energy of the electrons.

XLVI. A Note on Magneto-striction in Degenerate Electron Gas. By D. V. GOGATE, Physics Department, Baroda College, Baroda *.

Introduction.

THE assumption that the valence electrons of the atoms are free in a metal and their number is of the same order of magnitude as the number of atoms is found to be of great value in connexion with the electrical and magnetic properties of metals. The electrons, in this case, constitute an electron gas, degenerate in the sense of Fermi-Dirac statistics. It was first shown by Pauli that the paramagnetic susceptibility for the free-electron gas, when degenerate, comes out to be entirely different from that of Langevin's classical case.

The object of this note is to derive an expression for magneto-striction, on the basis of the free electron model, by considering the effect of magnetic field on degenerate electron gas. In article (1) Pauli's expression for paramagnetic susceptibility is derived from Boltzmann's formula for the degenerate case. In article (2) the increase in pressure of the degenerate electron gas is calculated from the increase in the null-point energy, brought about by the application of a magnetic field which causes some of the electrons to migrate to previously unoccupied higher energy levels or phase cells. The slight increase in the volume of the degenerate gas, which must occur if pressure is to be kept constant, is then determined. This immediately gives a formula for magneto-striction.

1. Paramagnetic Susceptibility.

The Boltzmann formula applied to the degenerate case works out to be

$$E_2 - E_1 = \frac{\hbar^2}{2m} \left(\frac{3N_2}{4\pi g V} \right)^{2/3} - \frac{\hbar^2}{2m} \left(\frac{3N_1}{4\pi g V} \right)^{2/3},$$

where N_1 and N_2 denote the number of electrons in energy states E_1 and E_2 respectively and V is the volume of the assembly. To apply this formula to the case of paramagnetism, let N_2 denote the number of electrons with their magnetic moments parallel to the magnetic field

* Communicated by Dr. D. S. Kothari.

and N_1 the number of electrons with moments anti-parallel. Then

$$\frac{h^2}{2m} \left(\frac{3N_2}{4\pi V} \right)^{2/3} - \frac{h^2}{2m} \left(\frac{3N_1}{4\pi V} \right)^{2/3} = 2\mu H,$$

where μ is the magnetic moment of the electron and $g=1$. Now, let $N_2 = N_1 + x$; then x will be given by

$$\begin{aligned} 2\mu H &= \frac{h^2}{2m} \left(\frac{3}{4\pi V} \right)^{2/3} N_1^{2/3} \left\{ \left(1 + \frac{x}{N_1} \right)^{2/3} - 1 \right\} \\ &= \frac{h^2}{2m} \left(\frac{3}{4\pi V} \right)^{2/3} N_1^{2/3} \cdot \frac{2}{3} \frac{x}{N_1}. \end{aligned}$$

Therefore

$$x = \frac{6\mu H m}{h^2} N_1^{1/3} \left(\frac{4}{3} \pi V \right)^{2/3}.$$

Hence the paramagnetic susceptibility k will be given by

$$k = \frac{\mu x}{VH} = \frac{6\mu^2 m}{h^2} \left(\frac{N_1}{V} \right)^{1/3} \left(\frac{4\pi}{3} \right)^{2/3}.$$

Putting $N_1 = \frac{N}{2}$, we get

$$k = 12 \frac{\mu^2 m}{h^2} \left(\frac{N}{V} \right)^{1/3} \left(\frac{\pi}{3} \right)^{3/2}.$$

2. Magneto-striction.

In the absence of a magnetic field the electrons occupy the lowest available cells (two electrons in each). The occupied cell of maximum energy corresponds to the energy value

$$\frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}.$$

Let us now consider the N electrons divided into two groups, one having their spins in a definite direction and the other in anti-parallel direction. The number in each group will be $N/2$ in the absence of a magnetic field, and the null-point energy of each group will be

$$E_0 = \frac{3}{10} \frac{h^2}{m} \frac{N}{2} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N}{2V} \right)^{2/3},$$

the total energy being

$$2E_0 = \frac{3}{5} \frac{h^2}{m} \frac{N}{2} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N}{2V} \right)^{2/3} \dots \dots \dots \quad (1)$$

In the presence of the magnetic field, one group will contain N_1 electrons and the other $N_2 = N_1 + x$, and their energies will be given by

$$E_{N_1} = \frac{3}{10} \frac{h^2}{m} N_1 \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3}. \quad \dots \quad (2)$$

and

$$\begin{aligned} E_{N_2} = E_{N_1+x} &= \frac{3}{10} \frac{h^2}{m} (N_1 + x) \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1 + x}{V} \right)^{2/3} \\ &= \frac{3}{10} \frac{h^2}{m} N_1 \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} \\ &\quad + \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} \cdot \frac{5}{3} x. \quad \dots \quad (3) \end{aligned}$$

Adding (2) and (3) we get the total energy

$$= \frac{3}{5} \frac{h^2}{m} N_1 \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} + \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} \cdot \frac{5}{3} x. \quad (4)$$

Comparing (4) with the expression (1) for total energy without a magnetic field we get:

Excess of energy

$$\begin{aligned} &= \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} \frac{5}{3} x \\ &= \frac{1}{2} \frac{h^2}{m} \left(\frac{3}{4\pi} \right)^{2/3} \left(\frac{N_1}{V} \right)^{2/3} \cdot \frac{6\mu H m}{h^2} N_1^{1/3} \left(\frac{4\pi V}{3} \right)^{2/3} \\ &= 3\mu H N_1 = \frac{3}{2} \mu H N. \end{aligned}$$

Now excess of pressure

$$\begin{aligned} &= \frac{2}{3} \text{excess of energy per unit volume} \\ &= \frac{\mu H N}{V}, \end{aligned}$$

and pressure p

$$\begin{aligned} &= \frac{2}{3} \cdot \frac{3}{40} \frac{h^2}{m} \left(\frac{3}{\pi} \right)^{2/3} \left(\frac{N}{V} \right)^{2/3} \cdot \frac{N}{V} \\ &= \frac{1}{20} \frac{h^2}{m} \left(\frac{3}{\pi} \right)^{2/3} \left(\frac{N}{V} \right)^{5/3}, \end{aligned}$$

$$\text{or } p = \frac{\text{const.}}{V^{5/3}} \quad \text{and} \quad \frac{dp}{p} = -\frac{5}{3} \frac{dV}{V} \quad \text{or} \quad \frac{dV}{V} = -\frac{3}{5} \frac{dp}{p}.$$

Hence

$$\frac{dV}{V} = \frac{3}{5} \frac{\mu H N}{V} \cdot \frac{20m}{h^2} \left(\frac{\pi}{3}\right)^{2/3} \left(\frac{V}{N}\right)^{5/3}.$$

(the +ve sign is taken here because dp has to be -ve if pressure is to be kept constant).

Hence, finally,

$$\frac{dV}{V} = 12 \left(\frac{\pi}{3}\right)^{2/3} \frac{m}{h^2} \frac{\mu H}{(N/V)^{2/3}} = \frac{K H}{n^{2/3}},$$

where

$$K = 12 \left(\frac{\pi}{3}\right)^{2/3} \frac{m}{h^2} \mu = 2.4 \times 10^7 \text{ approx.}$$

The applicability of this result to metals will be discussed in a subsequent communication.

In conclusion, I wish to express my sincere thanks to Dr. D. S. Kothari for his kind interest.

Baroda College,
25th June, 1936.

XLVII. *The Resistance to the Flow of Water along Smooth Rectangular Passages, and the Effect of a Slight Convergence or Divergence of the Boundaries.* By J. ALLEN, M.Sc., Assoc.M.Inst.C.E., and N. D. GRUNBERG, M.Sc. (University of Manchester) *.

List of Symbols.

a =half width of rectangular pipe.

b =half depth of rectangular pipe.

A =area= $4ab$.

m =hydraulic mean depth = $\frac{ab}{a+b}$.

p =pressure.

Q =volume flowing through pipe per unit time.

w =velocity of a particle parallel to the axis of the pipe.

w_{\max} =maximum velocity in cross-section.

y =coordinate measuring parallel to the depth.

* Communicated by the Authors.

x =coordinate of particle, taking the centre of the section as origin and measuring parallel to the width.

z =distance parallel to the longitudinal axis of the pipe.

μ =coefficient of viscosity.

ρ =density.

$\nu = \mu/\rho$ =kinematic viscosity. (The values assumed for ν were extracted from International Critical Tables, 1929.)

g =acceleration due to gravity.

$$h_T = \text{true kinetic head} = \frac{1}{2g} \iint w^2 dx dy.$$

$$h_a = \text{apparent kinetic head} = v^2/2g.$$

$$v = \text{mean velocity} = \frac{Q}{4ab}.$$

$$R = \text{resistance per unit area of walls} = -m \frac{dp}{dz}.$$

$$\text{Reynolds's number} = \frac{vm}{\nu}.$$

$$K = \frac{R}{\rho v^2} \cdot \frac{vm}{\nu}.$$

Δ =viscous loss of head.

δh =loss of head observed by means of differential gauge.

$$\tau = -\frac{1}{2\mu} \cdot \frac{dp}{dz}.$$

I. INTRODUCTION.

THE motion of a fluid through a pipe of rectangular cross-section is of considerable theoretical interest, because there is a fundamental difference between such passages and those having a circular section. Whereas all smooth, cylindrical pipes are geometrically similar to one another, rectangular passages must be considered in relationship to their ratio of width to depth.

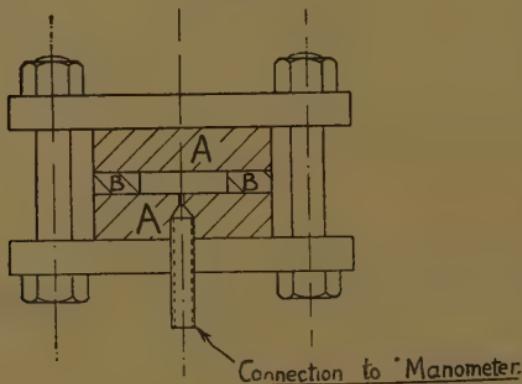
The present investigation was designed to extend the researches of previous experimenters on uniform rectangular pipes and also to determine the effect of a slight convergence or divergence on the resistance.

II. APPARATUS AND PROCEDURE.

The pipe was made up, as shown in the cross-section of fig. 1, of two rolled brass bars A, A, with brass strips B, B placed between them, the whole being tightly clamped together by means of a number of steel bars and $\frac{3}{8}$ -inch diameter bolts and nuts. The clamps were spaced every few inches along the 15 feet length of the pipe, and the surfaces of the side-strips B, B smeared with a very thin film of vaseline before clamping. The whole pipe was supported at a number of points on wooden blocks.

To facilitate the construction of the parallel or uniform channel, a steel template 1 inch wide $\times \frac{3}{16}$ inch thick

Fig. 1.



$\times 12$ inches long was drawn through the pipe after first lightly clamping; the side pieces were pressed against it and the clamps tightened. After two or three trials in this way it was found possible to obtain a uniform width within ± 1.5 per cent.

The method adopted for measurement of the dimensions was to determine by means of a micrometer the dimensions of the component bars or strips themselves at a number of points along their length, and also the outer dimensions of the finished channel. It is estimated that an error of 1 per cent. might result from this method of measuring the internal depth of the pipe; not more than $\frac{1}{2}$ per cent. in measuring the internal width. This question will be discussed further in Section III.

For the purpose of the measurement of pressure-drop, holes approximately 0.02 inch diameter were drilled in the lower plate and communicated with copper tubes screwed into the plate and soldered to make a joint. These tubes were connected to (a) an inclined water-air differential gauge read through a travelling microscope, or (b) a vertical water-air gauge, or (c) a water-mercury gauge, according to the pressure-difference to be measured.

The inclined gauge was frequently checked by connecting to two vessels containing water at a difference of level readable directly on vernier pointer gauges.

The gauge-length adopted was 152.4 cm., chosen centrally so that some 5 feet of pipe lay on either side of the experimental portion, not reckoning the piping leading from the supply reservoir to the inlet end and the pipe leading from the channel to the collecting vessels at the discharge end.

Water was supplied from an overhead gravity tank equipped with ball-valve to maintain a constant head, or alternatively, for three or four of the biggest discharges, direct from the mains.

The temperature of the water flowing through the pipe was determined by two calibrated thermometers, one near the inlet and one near the exit. These thermometers were secured by rubber tube to a short copper pipe in communication with the channel.

The rate of discharge was measured by means of calibrated collecting vessels and a stop-watch reading to tenths of a second; the shortest time for collecting a discharge was some 60 seconds.

Considering next the convergent pipe, the side-strips were first cemented on the outside at odd places to the bottom bar so as to give as near as practicable a uniform side convergence. The top bar was then lifted gently into position and the whole clamped together as before. Measurements of width and depth were taken, as before, by means of a micrometer screw-gauge.

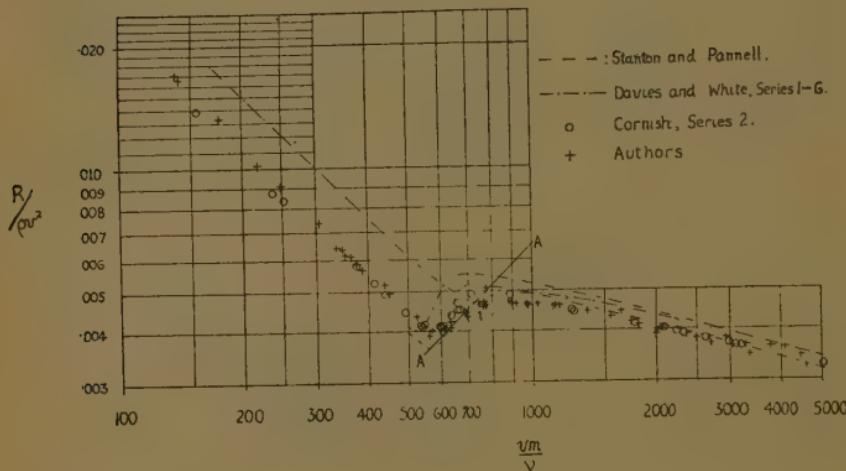
To obtain a divergent pipe, the inlet and outlet connexions of the convergent channel were reversed.

In the case of the convergent and divergent pipes, the lower pressure differences were measured on a calibrated toluene-water manometer; the gauge-length adopted was a central distance of 214.0 cm.

III. RESULTS OBTAINED WITH UNIFORM PIPE.

The dimensionless quantities $R/\rho v^2$ and vm/v are plotted logarithmically in fig. 2, together with results of other experimenters (Cornish *, Davies and White †) whose published papers give sufficient data for the re-plotting of their curves. The classical lines for cylindrical pipes as obtained by Stanton and Pannell ‡ are also shown.

Fig. 2.



The following table gives the dimensions of the rectangular pipes to which reference is made :—

Experimenters.	2a.	2b.	a/b.	m.	K.
Cornish	1.178	0.404	2.92	0.150	2.12
Authors	2.550	0.651	3.92	0.259	2.27
Davies and White (Series 1-6). } 2.54	{ 0.015 to 0.035	{ 169 to 72.7	{ 0.0075 to 0.0175 } 3.00		

(Dimensions in cm.)

In the last column of the above table the theoretical value of K is quoted corresponding to the equation

* Proc. Roy. Soc. A, cxx. p. 691 (1928).

† Proc. Roy. Soc. A, cxix. p. 92 (1928).

‡ Phil. Trans. A, cxciv. p. 199 (1914).

$R/\rho v^2 = K\nu/vm$. For the authors' pipe, the value of K is 2.27, assuming dimensions to be accurately determined; the average of the first 17 of their experimental observations gives $K=2.23$. The lines obtained for laminar flow by the other experimenters, using pipes of very different absolute dimensions and ratios of sides, as well as different gauge-lengths and various arrangements of apparatus, also agree reasonably with the appropriate theoretical laws; it is therefore established that the equations of laminar motion are quite generally confirmed by experiment within the limits of observation.

The "lower critical point" is seen to depend essentially upon the ratio of a to b , as previously suggested by Cornish, and, indeed, is closely obtained by producing the laminar line until it strikes the line marked AA in fig. 2, AA being perpendicular to the laminar line. The comparative values of the critical vm/ν thus estimated would be :

Experimenters.	a/b .	Critical vm/ν .	Theoretical ratio of $w_{\max.}$ to v of laminar flow.
Cornish	2.92	590	1.87
Authors	3.92	600	1.79
Davies and White....	$\rightarrow\infty$	730	1.50

It is of interest to note that these critical numbers are approximately inversely proportional to the ratio of $w_{\max.}$ to v in laminar motion.

If we assume the empirical law

$$(vm/\nu)_c \propto \frac{1}{r},$$

where

$$r = w_{\max.}/v,$$

and $(vm/\nu)_c \rightarrow 730$ as $a/b \rightarrow \infty$, i. e., as $r \rightarrow 1.50$,

then
$$(vm/\nu)_c = \frac{1100}{r}.$$

The critical numbers corresponding to various ratios of sides then become :

a/b .	r .	$(vm/v)c$.
1.0	2.10	525
2.0	2.00	550
3.0	1.86	590
4.0	1.79	615
5.0	1.74	630
∞	1.50	730

Considering next the resistance to turbulent flow, it is seen that there is close agreement between the authors' observations and Cornish's. Davies and White's, however, lie somewhat higher, as far as the range of (vm/v) covered in the tests. For moderate ratios of a to b , over the range exhibited in fig. 2,

$$R/\rho v^2 = 0.0247 (v/vm)^{0.24}.$$

According to Davies and White *, the corresponding law for cylindrical pipes is $R/\rho v^2 = 0.028(v/vm)^{0.25}$, while Allen † has found $R/\rho v^2 = 0.0117(v/vm)^{0.15}$ for open rectangular channels. The law for rectangular pipes is thus intermediate between that for cylindrical pipes and that for open rectangular channels, but much nearer to that for cylindrical pipes.

IV. RESULTS OBTAINED WITH CONVERGENT PIPE.

The measured dimensions of the pipe were as follows :

z (inches).	a (cm.).	b (cm.).	m (cm.).	a/b .	Remarks.
0....	1.021	0.324	0.246	3.11	First gauge-point.
12....	.942	.326	.242	2.89	
24....	.890	.324	.237	2.75	
36....	.792	.323	.230	2.46	
48....	.795	.323	.230	2.46	
60....	.743	.324	.226	2.29	
72....	.654	.320	.215	2.04	
84....	.616	.318	.206	1.94	Second gauge-point.

* "A Review of Flow in Pipes and Channels," "Engineering," July-August, 1929.

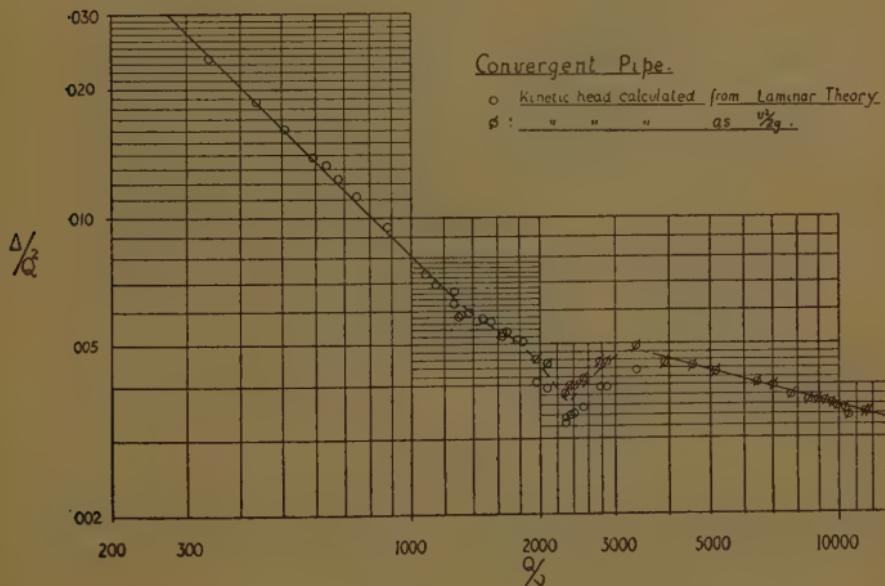
† "Streamline and Turbulent Flow in Open Channels," Phil. Mag. (7) xvii. p. 1081 (1934).

It will be observed from this table that the sides did not converge at a perfectly uniform rate; indeed, over a certain length they were approximately parallel. This is due to practical difficulties in fixing the side strips, and does not invalidate the experiment, which was intended to demonstrate the interesting effects of a slight general convergence. Furthermore, by the method of arithmetic integration explained in Appendix B, account is taken of the variations in section.

In Appendix A a theory is presented whereby the law of laminar motion through a slightly convergent or divergent pipe may be obtained. Applying this theory to the present case indicates that Δ/Q^2 should equal $8.11\nu/Q$.

Fig. 3 shows Δ/Q^2 plotted against Q/ν logarithmically :

Fig. 3.



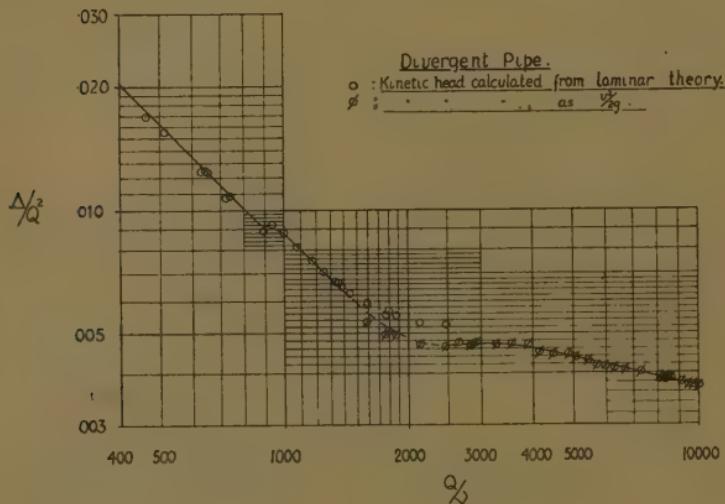
the line $\Delta/Q^2 = 8.11\nu/Q$ is drawn for comparison, and it is seen that the observations closely agree with it at the lower Reynolds's numbers, but depart from it appreciably in the region of $vm/\nu = 1500$.

A difficulty arises in the value to be assigned to the head necessary to provide the acceleration of the stream between the first and second gauge-points. It will be seen that in fig. 3 points are given different symbols

according as to whether the true kinetic head has been calculated, as demanded by the theory of laminar motion or otherwise, simply as $v^2/2g$. There can be little doubt that, once turbulence is established, the kinetic head approximates within a few per cent. to the apparent head $v^2/2g$: the error in assuming it to be actually $v^2/2g$ will be small in the final estimate of Δ/Q^2 .

Intermediate between true laminar and turbulent states, however, there is presumably a transition from kinetic head = "laminar value" to kinetic head = $v^2/2g$. A broken line is accordingly drawn in fig. 3 to join the

Fig. 4.



laminar and turbulent lines in such a way that the allowance for acceleration is made gradually. This broken line is, of course, an assumed curve, but it appears to give a reasonable interpretation in relation to the magnitude of the quantities involved.

V. RESULTS FROM DIVERGENT PIPE.

In fig. 4 the observed values of Δ/Q^2 and Q/v are plotted in similar manner to fig. 3. Again between the laminar and turbulent states a broken line is shown on the assumption of transition in the magnitude of kinetic head.

VI. COMPARISON OF CONVERGENT, DIVERGENT,
AND UNIFORM PIPES.

To demonstrate the comparison fig. 5 has been prepared, in which $R/\rho v^2$ is plotted against vm/ν logarithmically. In preparing this diagram the following process has been adopted :—

The theory of Appendix B shows that, for the given convergent or divergent pipe,

$$\Delta/Q^2 = 8.11\nu/Q, \text{ or } g.dp/Q = 8.11\mu,$$

an equation which is, in fact, closely confirmed by the experiments.

If the pipe were of uniform section, width $2a$, depth $2b$,

$$Q \text{ would} = -\frac{4ab^3}{3\mu} \frac{dp}{dz} \left\{ 1 - \frac{192}{\pi^5} \frac{b}{a} (\tanh \pi a/2b + \frac{1}{3^5} \tanh 3\pi a/2b + \dots) \right\}$$

Taking $b=\text{constant}=0.323$ cm., and comparing this equation with

$$Q = \frac{g.dp}{8.11\mu},$$

shows (by trial and error) that the equivalent uniform section to give the same Q for the same pressure-drop between gauge-points (1) and (2) is

$$a=0.795, \quad b=0.323 \text{ cm.}$$

$$\text{But } R/\rho v^2 = \frac{\Delta}{Q^2} \left\{ \frac{gA^2m}{\rho.dz} \right\},$$

$$vm/\nu = \frac{Q}{\nu} \cdot \frac{m}{A},$$

where $A=4ab$

$dz=\text{gauge-length from (1) to (2).}$

Accordingly, figs. 3 and 4 have been translated into a form suitable for fig. 5 by multiplying the ordinates,

Δ/Q^2 , by $\left(\frac{gA^2m}{\rho dz}\right)$ and the abscissæ, Q/ν , by m/A .

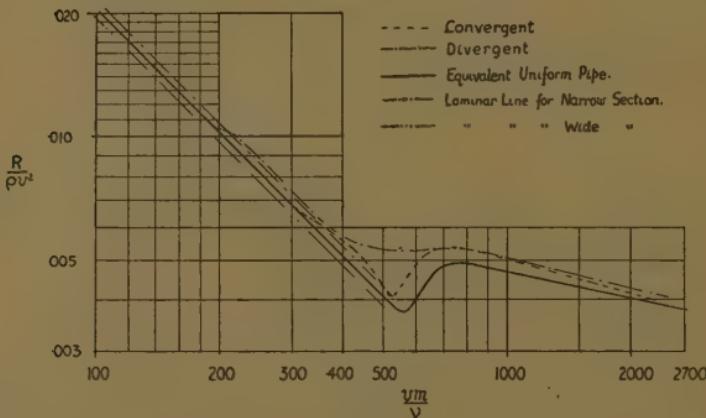
Again, the curve for a uniform section, $a=0.795$ and $b=0.323$ cm., based on fig. 2, is plotted in fig. 5 for comparison.

Further, the laminar lines are shown corresponding to the sections at the gauge-points.

Examination of fig. 5 reveals, then, that :

(a) With either slight convergence or divergence there is a point of breakdown relatively early, but much earlier with divergent than convergent flow. Thereafter the flow appears to be approximately re-stabilized for a time to follow the law corresponding to the narrower end-section.

Fig. 5.



(b) With divergent boundary, there is a region ($vm/v=400$ to 600) in which the resistance is considerably greater than that of a convergent or a uniform channel. This, presumably, is due to the accepted phenomenon of the essential instability of divergent flow resulting in eddy-formation. This effect persists into the turbulent region where the eddy loss due to divergence is superimposed upon the loss due to friction.

(c) On the other hand, the turbulent line of the convergent pipe appears to be rapidly approaching that of the equivalent parallel pipe.

VII. SUMMARY.

The experiments made by Davies and White, Cornish, and the authors confirm the theory of resistance in

laminar motion through rectangular pipes of various sections so long, of course, as the flow is of the nature visualized in the theory.

The lower critical velocity is, within experimental errors, inversely proportional to the ratio of maximum to mean velocity in the cross-section during laminar flow.

For moderate ratios of side dimensions (Cornish 2.92, authors 3.92), the law of resistance in the turbulent region $vm/v = 1000$ to 5000 is closely given by

$$R/\rho v^2 = 0.0247 (v/vm)^{0.24}.$$

The theory presented in Appendix A for slightly convergent or divergent boundaries is confirmed by experiment at sufficiently low Reynolds's numbers (as defined in figs. 3 and 4). Points of breakdown occur, however, followed by a period of approximate adherence to the law corresponding to the narrow end of the channel (fig. 5).

Even the slight divergence of the authors' channel is found to promote eddy-loss which appreciably increases the resistance both in the transition state and after the establishment of "fully turbulent" conditions (fig. 5).

The slight convergence, however, gives results in the turbulent region which approach those of the equivalent uniform section (fig. 5).

The authors wish to express their sense of indebtedness to Professor A. H. Gibson, D.Sc., LL.D., for the facilities placed at their disposal in the Whitworth Engineering Laboratories of Manchester University.

APPENDIX A.

Theory assumed for Laminar Flow through slightly Convergent or Divergent Pipe of Rectangular Section.

Assuming the streamlines to be sensibly normal to any cross-section, and dividing the pipe into elementary lengths dz , dp = pressure-drop due to viscous resistance

$$= - \frac{3Q\mu}{4ab^3} \frac{dz}{\left[\left(1 - \frac{192a}{\pi^5 b} \tanh \pi a/2b + \frac{1}{3^5} \tanh 3\pi a/2b + \dots \right) \right]}.$$

Hence, between gauge-points (1) and (2),

$$p = -\frac{3}{4}Q\mu \int_{z_1}^{z_2} ab^3 \frac{dz}{\left[1 - \frac{192}{\pi^5} \frac{b}{a} (\dots \dots) \right]}.$$

$$= -\frac{Q\mu}{4} \int_{z_1}^{z_2} \frac{(a+b)^2 K dz}{a^3 b^3}.$$

If, then, the quantity $\frac{(a+b)^2}{a^3 b^3} K$ be plotted against z ,

and the curve integrated by planimeter giving an area A' , the viscous loss between the gauge-points may be estimated as

$$p = -\frac{Q\mu}{4} A'.$$

In the case of a convergent pipe, however, a pressure drop must also accompany the acceleration of the stream ; in a pipe which is divergent in the direction of flow a pressure rise will result from retardation.

Let

h_{T_1} = true kinetic head at gauge-point 1 ;

h_{T_2} = , , , , , 2 ;

δh = total loss of head observed on a differential gauge connected to (1) and (2) ;

Δ = viscous loss of head ; then

$$\Delta = \delta h - (h_{T_2} - h_{T_1}).$$

For different ratios of a/b , h_T may be obtained in terms of $v^2/2g$ by reference to a paper previously published †.

* $K = \phi(a/b)$. The curve representing this function has been drawn by Lea and Tadros, Phil. Mag. ser. 7, vol. xi. p. 1247 (1931). It has also been calculated and plotted to a large scale by the authors for use in applying the theory of Appendix A to their convergent and divergent pipes.

† Allen, Phil. Mag. (7) xxii. p. 249 (1936).

APPENDIX B.

Typical Experimental Observations.

Uniform Pipe.

Water (temp. ° C.).	Q (c.c./sec.).	δh (cm. of water).	$R/\rho v^2$.	$vm/v.$	K.
11.13	11.22	.466	.0170	138	2.34
11.08	11.43	.474	.0166	140	2.32
9.95	14.80	.630	.0132	176	2.32
8.01	19.38	.832	.0101	218	2.20
9.34	21.22	.895	.00908	247	2.24
9.27	26.32	1.112	.00739	305	2.25
8.44	29.39	1.207	.00640	335	2.15
9.67	29.32	1.186	.00634	345	2.19
7.84	31.71	1.341	.00613	351	2.15
10.00	30.42	1.230	.03609	364	2.22
8.55	32.84	1.380	.00586	375	2.20
10.13	32.23	1.280	.00565	385	2.17
8.84	37.75	1.625	.00521	435	2.27
8.16	39.79	1.710	.00495	447	2.21
7.51	46.88	2.07	.00434	522	2.26
7.75	49.42	2.078	.00390	558	2.18
8.53	49.67	2.15	.00400	567	2.27
7.97	56.43	2.71	.00405	630	
6.83	58.51	3.15	.00421	630	
8.04	60.89	3.59	.00443	689	
7.26	63.28	3.78	.00432	694	
12.16	59.37	3.52	.00458	759	
7.33	67.31	4.58	.00464	730	
10.21	61.92	3.56	.00426	742	
10.04	61.83	3.67	.00441	741	
7.25	69.37	4.91	.00469	760	
7.61	69.99	4.94	.00462	779	
9.70	65.67	4.77	.00506	779	

XLVIII. *The Diameter of the Lightning Channel.* By Prof. B. F. J. SCHONLAND, *The Bernard Price Institute of Geophysics, Johannesburg* *.

[Plate IV.]

1. **A**LTHOUGH theoretical studies of the conditions prevailing in the channel of the lightning discharge introduce the diameter of the channel as an important parameter, it would appear that experimental values

* Communicated by the Author.

for the actual diameter are both uncertain and conflicting. Thus Toepler* gives an upper limit of 40 cm. for the diameter, but remarks that the separations of the oppositely magnetized regions of basalt blocks struck by lightning frequently indicate diameters of a few centimetres only.

Evidence of this kind is rather indirect and a variety of factors may lead to error. Small diameters can arise if the object examined has received only a portion of the discharge current, as a result of branching of the flash near the ground. Large diameters can be wrongly inferred if the separate strokes of the discharge have had their channels shifted by the wind in the intervals between them.

This effect of the wind makes ordinary photographs taken on a fixed camera unsuitable for finding the channel diameter. It can be avoided by moving the camera so as to separate the stroke images completely from one another. For such a photograph to be of value, however, it is necessary for the angle subtended by the channel diameter at the lens to be greater than the minimum angle the lens can resolve, otherwise the diameter observed is constant and set entirely by diffraction. This means in practice that the photograph must be taken at a fairly short distance from the discharge, a requirement which is not easy to fulfil.

The present paper is concerned with a photograph in which the requisite shift in the position of the images of successive strokes was actually produced by the action of the wind itself. It was taken on a fixed camera at a short distance from the discharge by Mr. C. Dolbey, of the Victoria Falls and Transvaal Power Company, Johannesburg.

2. The photograph shown in Pl. IV. was taken with a firmly fixed Dallmeyer quarter-plate reflex camera of focal length 15 cm., stopped down to an aperture of f/11. The camera was facing due north and the flash occurred to the N.N.E. No rain was falling at the time, though a heavy downpour occurred a few minutes later. The wind velocity at the camera was between 20 and 30 miles an hour towards the N.E. The discharge partly stunned the observer, and a "persistent sulphurous

* M. Toepler, *Hermsdorf-Schomburg Mitt.* xxv. p. 743 (1926).

odour" was apparent for some time after it occurred. The picture was taken on Kodak super-sensitive panchromatic film.

The fact that the camera was firmly fixed at the time the photograph was taken is established by the clear outline of the fixed objects in the foreground, while the outlines of the two small trees immediately in front of the house are blurred because of their movement in the wind during the discharge.

The total duration of the discharge appeared to the observer to be only a fraction of a second. An estimate of the actual duration, based upon the photograph itself, is given later. The important question of the distance between discharge and camera has been carefully investigated by Mr. Dolbey, who has found that the flash struck rocky ground within a few hundred feet of a low range of hills at a distance from the camera lying between 230 and 280 metres.

3. The photograph shows the discharge to have been shifted by the wind in such a manner as to separate out the channels of eleven component strokes. The first stroke channel lies to the left since the component of wind velocity perpendicular to the line of sight was directed to the east. This first stroke gave rise to five branches and was considerably brighter than any succeeding stroke. In the case of strokes 2, 3, and 4 and strokes 8 and 9 the separation produced by the wind was only just sufficient to show their channel images resolved.

4. Examination of the channels of successive strokes shows that marked variations exist in the details of kinks and bends, variations which indicate that there were considerable local fluctuations in wind velocity and direction. There are, however, no important large scale variations to be observed in the wind over the 70 metre length of discharge path photographed. The separation of strokes 1 and 11 decreases by 30 per cent. between the bottom and the top of the portion examined, and the direction of the wind, as shown by the arrows drawn in the figure, is practically horizontal throughout.

5. Direct measurement of the photograph at the lower end of the discharge, combined with the known distance of the discharge, shows that the actual separation of the

first and last channels in the discharge was 3.7 metres perpendicular to the line of sight.

From the data given above we may take the wind velocity in this direction to have been 29 km./hr. or 8 metres/sec., whence the time interval between the first and last stroke was .46 seconds. The average time interval between strokes was thus .046 sec., in satisfactory agreement with the most probable value for this interval, .03 sec., previously found for storms in this locality *. As a useful consequence of this agreement it follows that few if any of the separated channels photographed correspond to more than one stroke of the discharge.

The longest time interval between successive strokes is that between the two last, namely, .08 sec. The shortest interval, in the cases of the combinations 2, 3, 4 and 8-9, is about .02 sec.

6. In order that the diameter of the photographic image may correctly represent the actual diameter the angle subtended by the latter at the lens must exceed $1.22 \lambda/D$, where λ is the effective wave-length of the light and D is the diameter of the lens aperture. Thus the actual width of the channel must exceed $l \times 1.22 \lambda/D$, where l is the distance of the flash. Substituting $\lambda = 6 \cdot 10^{-5}$ cm., $D = 1 \cdot 36$ cm., and $L = 2 \cdot 5 \cdot 10^4$ cm., it is found that this width must exceed 1.3 cm. The actual widths obtained in the next section for the lightning stroke channels exceed this critical minimum by factors greater than 10.

7. To obtain the actual diameter we multiply the image widths by $2.5 \times 10^4/15$ and obtain the results shown below.

Stroke Channel Diameters.

Stroke No.....	1.	2, 3, 4. (av.)	5.	6.	7.	8-9. (av.)	10.	11.	Branch X. (top.)	X. (btm.)	Y.	Z.
Diameter (cm.)	23	20	16	16	16	15	19	19	16	11	13	17

The values range from 23 cm. for stroke 1 to 15 cm. for 8 and 9, and still lower figures are found for the branches.

* Schonland, Malan, and Collens, Proc. Roy. Soc. A, clii. p. 595 (1935).

These results therefore indicate that the channel diameter is about 16 cm., increasing somewhat in the case of heavy first strokes and falling in the case of branches.

If over-exposure and halation were present on these records the above values would be upper limits to the diameter. A careful examination of the original negative, however, shows that the flash was not over-exposed, while no large changes in diameter accompany the very different intensities of the different strokes.

8. The values obtained above for the diameter of the lightning channel are, so far as I am aware, the first which can claim to be free from ambiguity in respect of overexposure and partial shift of the channels of separate strokes. It is of interest to compare them with the corresponding diameters of heavy current spark discharges in air.

Bellaschi *, using currents of the same order as those in the lightning discharge, 14,000 amperes, reports photographically determined diameters of 10-25 cm. which are of the same order as those found above. In his opinion these values, however, considerably exceed the true diameter of the electrical "core" of the discharge, and he describes two experimental investigations which are held to establish this view. Observations of the burning produced on the surface of a polished electrode indicate 1.5 cm. for the diameter, while others of the pressure developed in the course of passage through a fibre-tube show that if the diameter of the tube is less than 2 cm. explosive pressures exceeding 15,000 atmospheres are developed.

It would seem that these two experiments are conflicting in their results, for since the temperature of the channel of the spark discharge is unlikely to exceed 12000° , and since the density of the air must remain constant for the short time involved in the heating process, the maximum pressure to be expected in the channel is easily shown to be 40 atmospheres. It would thus seem that the very much higher pressures found by Bellaschi in fibre-tubes of 2 cm. diameter were due to the confinement of the discharge to channels much smaller than usual. The argument therefore suggests that the diameter of the electrical core of the discharge is much larger than 2 cm.,

* Bellaschi, Trans. Am. I.E.E. (1935).

and may well be of the same order as that of the luminous sheath examined by the camera, namely, 16 cm. or more.

This suggestion conflicts with the result of Bellaschi's first experiment, but it seems possible that the interpretation of this experiment may be in error. It involved the determination of the channel diameter at the electrode itself, and an artificial "leader" in the form of a fine wire was used to carry the discharge to the centre of the polished electrode.

My thanks are due to Mr. C. Dolbey for his kindness in placing this photograph at my disposal and for his help in finding the distance and other factors involved in the discussion.

Summary.

A lightning photograph is discussed which shows the first and last separate strokes of the discharge to have had their channels shifted a distance exceeding 3·7 metres by the wind. It is found that the average diameter of the lightning stroke channel is about 16 cm. The heaviest stroke gave a diameter of 23 cm. and the weakest branch one of 11 cm. Consideration is given to the question whether this diameter represents that of the electrical core of the discharge.

XLIX. Notices respecting New Books.

The Scientist in Action. By W. H. GEORGE. [Pp. 335.] (Williams and Norgate, 1936. Price 10s. 6d.)

THE keynote of Dr. George's study of scientific method is that science is inseparable from scientists but quite separate from philosophy. After a general discussion of the characteristics of "the scientific outlook" there is a detailed consideration of the scientist in action in "getting scientific facts" and "arranging scientific facts." As befits "a scientific study" of the scientist's method the treatment is essentially descriptive; but the description is no more "mere" description than are those in science generally. In unravelling the essentials of so complex an activity penetrating insight and critical judgment are called for no less than in dealing with a specialized scientific problem.

Any such view as that the products of scientific research are fragments of Absolute Truth which the investigator has laboriously ferreted out is, of course, completely irrelevant, if not meaningless, from the standpoint adopted in this book. The author rightly objects, and very strongly, to the presentations characteristic of many popular books on science in which the science is inextricably mingled with speculative metaphysics. It is to be hoped that Dr. George's book will be widely read by that public, seriously but non-professionally interested in science, who may well have gained erroneous impressions. To most professional scientists, however, Dr. George's main theme will appear less novel. Indeed, his central thesis may almost be described as completely orthodox. The author gives evidence of wide and appreciative reading on the biological and psychological as well as the physical side of science ; and there are many apt quotations. The omission of reference to one of the most brilliant (if lengthy) presentations of the essentials of his own point of view—in Campbell's ' *Physics : the Elements* '—is therefore the more surprising. For Dr. George, moreover, the basic facts of science are coincidence observations ; and the analysis of the procedure in getting these facts is admirable ; but Eddington must indeed have fallen out of favour in that there is no reference to his treatment of this theme, elaborating his famous statement, " The whole subject matter of exact science consists of pointer readings and similar indications."

One of the many interesting features of Dr. George's treatment is his description of the scientist's arrangement of facts in terms of a " patterning " activity. The making of patterns, completed wholes, is seen as characteristic of many forms of human activity. The importance of pattern properties, which cannot be seen as inherent in the parts but which are characteristic of wholes, is emphasized, and illustrations are drawn not only from science, as in connexion with crystallography and structural chemistry, but also from literature and music. The characteristics of those patterns which are scientific theories are well discussed under eleven main headings.

The later part of the book deals with some factors in experimental technique and the future of experimental research. Here Dr. George discusses well such matters as the scientific usage of the " cause and effect " terminology ; and questions of the difficulty and cost of research (some rather surprising estimates are given), pioneering and " pump-handle " research, and the teaching of scientific method. Here it may be suggested that an attempt to deal with the general characteristics of scientific method in science teaching in universities is by no means so uncommon as the author apparently thinks. The reviewer's impression, gained from conversations with his

colleagues, is that many university science teachers do attempt to bring out general characteristics of scientific method in dealing with particular exemplifications, firmly, but quite mistakenly, believing themselves to be almost unique in this respect.

At the end of the book the author gives a "personal explanation" and a summary of his main conclusions, both of which in a book of this kind are of great value to the reader.

Dr. George appears to react to the word "philosophy" almost as others do to the word "communism." This may well irritate some readers, but it does not detract from the value of the book, for it has resulted in a book which will clarify the whole position as to the character and status of science for the philosopher no less than the scientist. The close relation between scientific and other human activities is clearly brought out.

The book is well and interestingly written, lighted by flashes of humour and penetrating comment. It certainly may be most strongly commended both to the professional and non-professional scientific reader.

Geometry of Time and Space. By A. A. ROBB. (Cambridge University Press, 1936. Price 21s. net.)

THIS book, as the author points out, is essentially a second edition of that published by him in 1914 under the title 'A Theory of Time and Space.' It contains a good deal of new matter, some of the original matter has been re-written and simplified, and the change in title gives a fairer indication of the true nature of the work—the formal construction and investigation, on the basis solely of *conical order*, of Minkowskian four-dimensional geometry.

In his introduction and towards the end of the book Dr. Robb maintains with considerable vigour his objections to the relativist or Einstein view of simultaneity, and his view that ideas of order are more fundamental than ideas of measurement, and that the conception of measurement should be based on and constructed from ideas of order. To show that this is possible in the case of a Minkowskian four-dimensional space-time is the main purpose of his book.

To many of his readers, however, the chief interest of the book will be in Dr. Robb's geometrical system, his manner of founding it on conical order, and his enumeration and proof of its properties. It may be described as being substantially equivalent to the geometry of a real four-dimensional flat space Σ , which would be obtained if parallelism were defined by an ideal "solid at infinity" M , and perpendicularity and angle by means of a *real* absolute quadric surface Ω in M ,

whose equations, using homogeneous coordinates, with $u=0$ as the equation of M , would be

$$x^2+y^2+z^2-t^2=u=0.$$

The innovation, as compared with Euclidean geometry of three- or four-dimensional space, consists in the *reality* of the absolute element Ω ; and this leads to an unusual variety of different types of real lines, planes, and solids. In fact a line of Σ is an "optical line," an "inertia line," or a "separation line" according as it insets M in a point on, outside, or inside Ω , with similar distinctions for planes and solids; and there is generally not one but a number of analogues in this geometry of any metrical proposition of ordinary space, depending on the optical, inertia, or separation, character of the data. For example, a "separation plane" is one whose two points of intersection with Ω are imaginary; and in it, therefore, geometry is Euclidean, and Pythagoras's theorem or any similar theorem holds. But in an optical or inertia plane modifications are necessary.

Every point P of Σ is the vertex of a *real* "isotropic" three-dimensional quadric cone $P(\Omega)$, analogous to the familiar *imaginary* isotropic cone in ordinary space. Each such cone divides the space Σ into three compartments, two "interior" compartments, α and β , containing their boundaries and meeting at the vertex P , and the region γ exterior to the cone. Dr. Robb's idea then is that in such a space there can be said to exist among the points a *conical order*, such that if Q is any point of Σ distinct from P , then Q is *after* P if it belongs to the compartment α of P ; Q is *before* P if it belongs to β , and it is *neither before nor after* P if it lies in the exterior region γ of P . It is from this concept of conical order that Dr. Robb proceeds, like Euclid, by postulate, axiom, and theorem, to develop and investigate his geometry of space-time. The work makes solid reading; but the preliminary explanation lightens the initial shock; and in demonstrating the sufficiency of conical order as a basis for his system Dr. Robb has made a real contribution to the foundations of space-time geometry.

Differential and Integral Calculus.—Vol. II. By R. COURANT.
[Pp. 682 & x.] (Blackie. Price 30s. net.)

THE method of presentation in this book is similar to that of volume I., which appeared two years ago. Many of the chapters have appendices in which appear valuable additions to the subject-matter of the chapters, and especially the more difficult portions of the subject, such as the discussions of fundamental ideas. In this way the needs of two types of reader are satisfied—of the reader whose main purpose is to

proceed to applications and of the reader who wishes to study the subject more thoroughly and who requires a reasonably rigorous treatment.

While volume I. dealt mainly with the theory of functions of one variable, the second volume includes chapters on vector analysis, functions of several independent variables, differential equations, calculus of variations and functions of a complex variable (of these the two last-mentioned chapters have been specially written for the English edition), and a supplement on real numbers.

The methods developed are well illustrated by examples in applied mathematics and physics. Numerous examples for solution are suggested in the text, and over fifty pages are devoted to answers and hints. A useful index completes a very attractive volume.

Dynamics of Rigid Bodies. By W. D. MacMillan. [Pp. 478 & xiii.] (McGraw-Hill Co., Ltd. Price 36s.)

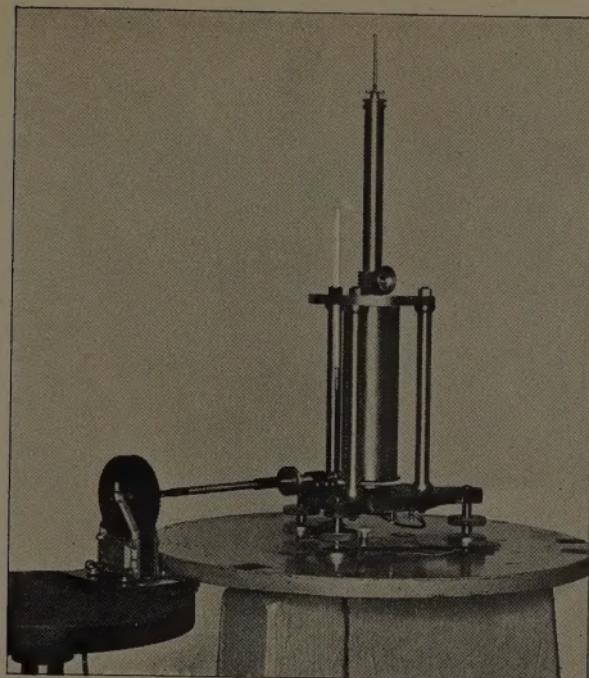
WITH this volume Professor MacMillan completes his series of books on mechanics, of which the first and second dealt respectively with statics and dynamics of a particle and with potential theory. Throughout the series both the geometrical and analytical methods of development have been used, and this treatment enables interpretation to be given to what would otherwise be somewhat meaningless mathematical formulæ.

The introductory chapters on vector algebra and moments of inertia are followed by discussions of systems of free particles, general theorems on the motion of a rigid body, problems of one and two degrees of freedom, the motion of a rigid body in space, and detailed considerations of the special problems of Euler, Lagrange, and Kowalevski. The equations of Lagrange and Hamilton are introduced in later chapters, and the last chapter discusses the method of periodic solutions and includes an account of the Bartky theory of linear differential equations with constant coefficients which involves the theory of matrices.

This book will be particularly useful to students reading an honours course in mathematics. A number of problems are given at the end of each chapter, and a list of selected references appears at the end of the book, together with author and subject indices. A point of adverse criticism should be mentioned—it is unfortunate that the volume is so highly priced.

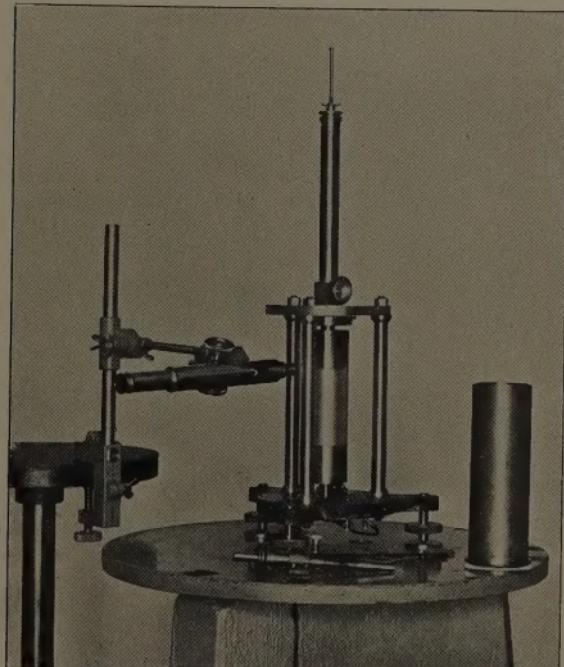
[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

FIG. 1.



Photograph of the apparatus with cover removed.

FIG. 2.



Adjustment of the inner cylinder; the outer cylinder removed.



Diameter of the lightning discharge.

